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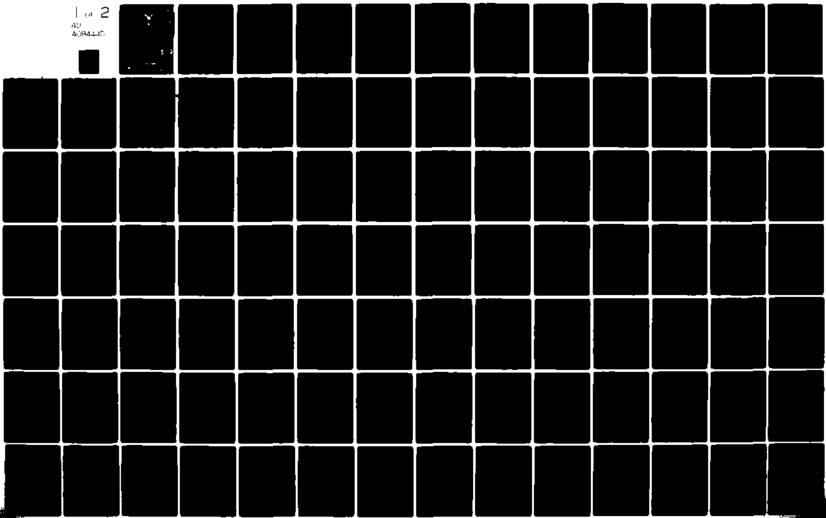

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The Application of Two Dimensional

Moment Invariants to

Image Signal Processing and

Pattern Recognition

THESIS

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THE APPLICATION OF TWO DIMENSIONAL MOMENT INVARIANTS TO IMAGE SIGNAL PROCESSING AND PATTERN RECOGNITION

THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
Air University
in Partial Fulfillment of the
Requirements for the Degree of
Master of Science

by

Tyle T. Kanazawa, B.S.

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Graduate Electro-Optics

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Preface

The study and research culminating in this thesis has been truly unique in my educational experience. It has expanded my perception and appreciation of the boundless horizons of physics. My deepest appreciation goes to my thesis advisor, Dr. Donn Shankland, whose invaluable insight provided a wall which my thoughts and ideas could be bounced on to see if they were valid.

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<u>Notation</u>

M_{pq}	pq-th raw moment of order (p+q)
μ_{pq}	pq-th central moment of order (p+q)
$\binom{a}{b}$	Binomial coefficient $\binom{a}{b} = \frac{a!}{b!(a-b)!}$
$\lvert a-b \rvert$	Absolute value of a-b
$(a_{po};\cdots;a_{op})(u,v)^p$	A homogeneous polynomial of variables u and v with coefficients $a_{po},\cdots,a_{op}$. Is the same as $a_{po}u^p + \binom{p}{1} a_{p-1,1}u^{p-1}v\cdots \binom{p}{p-1}a_{1,p-1}\, uv^{p-1} + a_{op}v^p$
Δ	Determinant of a matrix
G	A group, either abstract or transformation
$\bar{X}$	Vector
$\underline{X}$	Matrix
D(G)	Operator group representation of group G
D(R)	Operator corresponding to element R in group G
$\underline{D}$(G)	Matrix representation of the group G
δ_{ij}	Kronecker Delta Function
$D_{ij}(R)$	ij-th element of the matrix representation corresponding to element R in group G

$\chi(R)$	Character of element R of group representation D. $\chi(R)$ = trace of the matrix representation
O_T	An operator associated with transformation T such that if $\overline{X}' = T(\overline{X})$, then $O_T\,\Psi(\overline{X}') = \Psi(\overline{X})$
$(\overline{X},\overline{Y})$	Scalar product of vectors $\overline{X}$ and $\overline{Y}$
$(\cdot)^*$	Complex conjugate of a quantity
m_{ij}	ij-th element of matrix $\underline{M}$
$\underline{M}^\dagger$	Adjoint of $\underline{M}$
$\lvert\overline{X} - \overline{Y}\rvert$	Distance between vectors $\overline{X}$ and $\overline{Y}$
$\approx$	Equivalent representations
$\overline{D}(R)$	Adjoint representation
$\tilde{\underline{M}}$	Transpose of a matrix
$D^*(R)$	Complex conjugate representation
C_n	COS $n\theta$
S_n	SIN $n\theta$
$E(\cdot)$	Expected value operator $E(\cdot) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\cdot)\ \rho(x,y)\ dxdy$
i	$\sqrt{-1}$ imaginary number
$\underline{0}$	Zero matrix

Abstract

This thesis investigates the application of two-dimensional moment invariants to image pattern recognition. The general problem studied is how to identify an aircraft target and its orientation in real time. The method of moment invariants provides a clever feature extraction technique to reduce the information in an image to a finite number of quantities which are translation, size, and rotation independent. Most of the previous work on image pattern recognition has been based on the results obtained by M. K. Hu, who relied on the theory of algebraic invariants. In this thesis, a set of moment invariants is derived from the group-theoretical properties of the two-dimensional rotation group applied to the moments of an image intensity function. It is shown that Hu's invariants can be obtained from this set and is, in fact, an equivalent complete description of the image. The application of group methods to moments presents a general procedure for calculating moment invariants under any linear transformation. The image signal effect of thresholding the background clutter is also discussed.

I. Introduction

Artificial intelligence has long been an area of much study, and the ability to recognize visual images represents a major effort. However, research on pattern recognition has been hampered by the lack of a general theory of feature extraction. That is, what properties make one image unique from another, and how are those properties ranked according to their importance in the identification of the image? Generally, studies in pattern recognition have relied on heuristic criteria for feature selection. The method of moment invariants developed by M.K. Hu (Ref 1: 179-187) provides a clever feature vector with which an image may be described. This thesis further investigates the application of two-dimensional moment invariants to image signal processing to identify a three-dimensional object and its orientation.

The general problem formulation is how to efficiently utilize in real time visual information signals from an optical system for the identification and analysis of the visual pattern. Examples of the optical systems in question may include television monitors, infrared sensors, or even laser radar returns. Usually, the information content of even a single image is orders of magnitude greater than what may be actually needed to recognize the object. The method of moments provides a systematic procedure for

extracting numerical features from an image. By compiling a "library" of moment invariants for an object set, it may be possible to divide the feature vector space into separable regions by some type of classifier.

II. Previous Development

The original concept of using moment invariants for the purpose of visual pattern recognition was first published by M.K. Hu in 1962 (Ref 1:179 - 187). A framework was established for deriving a complete set of two-dimensional moment invariant functions under translation, rotation, and similitude. His approach was related to the study of algebraic invariants based on the work of the nineteenth century mathematicians Boole, Cayley, and Sylvester. Interest declined in the study of invariant forms following the initial nineteenth century work, but the application to visual pattern recognition has generated much recent interest.

S.B. Dudani extended the moment invariant concept to the identification of three-dimensional objects in his masters thesis (Ref 2) and in his doctoral dissertation (Ref 3) on an experimental study of aircraft identification using moment methods. Dudani's work formed the starting point of the U.S. Navy's algorithms in their Automatic Aimpoint Selection and Maintenance (AUASAM) program and also the U.S. Air Force's Image Processing Automatic Acquisition Control System (IPAAACS) conceptual design study (Ref 4:45). Dudani oriented his study toward video imagery and only used the information contained in the second and third order moments calculated over the image silhouette and boundary. After constructing classifiers (Bayes, K-nearest neighbor, sequential) over an object set, it was shown that the

classifier's performance was superior to human test subjects.

More recently, M.R. Teague established a framework for describing an image in terms of a finite number of moments (Ref 4). The inverse problem of how to reconstruct an image given a finite number of moments was addressed and served to illustrate the information content of successively higher order moments. For example, the second order moment approximation to an image intensity function is equivalent to an ellipse of constant intensity magnitude. A set of moments and moment invariants was also derived based on the orthogonal Zernike polynomials. In another paper, Teague described an optical processing scheme to obtain image irradiance moments of arbitrary order (Ref 5). The optical wave limitations on accuracy was also discussed.

Recently, Texas Instruments used moment invariants as part of the feature vector for a guidance algorithm in a demonstration at Eglin A.F.B. The algorithm was slated for use in a mini-computer for real-time performance (Ref 6).

III. Definition and Theory of Moment Invariants

Historically, M.K. Hu first conceived the idea of using moment invariants for pattern recognition of visual imagery (Ref 1: 179-187). His approach will be followed in this section. The moment invariant functions are related to the algebraic invariants, and a complete set of invariants is derived under translation, rotation, and similitude.

Raw and Central Moments

The concept of moments is not new, being used extensively in classical mechanics and statistical studies. The <u>two-dimensional (p+q)th order moment</u> of a density distribution function $\rho(x,y)$ in Cartesian coordinates is defined as

$$M_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p\, y^q\, \rho(x,y)\, dxdy \quad p,q = 0,1,2.... \tag{1}$$

For image processing purposes, $\rho(x,y)$ is the image intensity distribution across the optical plane. Under practical conditions all orders of moments exist, and the sequence of moments $\{M_{pq}\}$ is unique to $\rho(x,y)$, and conversely. These conditions are usually met, or at least approximated, over an optical system image plane.

The <u>central moment</u> μ_{pq} is defined as

$$\mu_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \bar{x})^p (y - \bar{y})^q \rho(x,y) \; d(x - \bar{x}) d(y - \bar{y}) \tag{2}$$

where

$$\bar{x} = \frac{M_{10}}{M_{00}} \qquad \bar{y} = \frac{M_{01}}{M_{00}} \; . \tag{3}$$

By a simple change of variables, it is obvious that under the transformation of translation,

$$\begin{aligned} x' &= x + \alpha \\ y' &= y + \beta \end{aligned} \tag{4}$$

where α and β are constants, the central moments are invariant.

From the definition of central moments, Eq (2), it is easy to express the central moments in terms of the raw moments. For example,

$$\begin{aligned} \mu_{00} &= M_{00} & \mu_{20} &= M_{20} - \bar{x}^2 M_{00} \\ \mu_{01} &= M_{01} - \bar{x} M_{00} = 0 & \mu_{11} &= M_{11} - \bar{x}\bar{y} M_{00} \\ \mu_{10} &= M_{10} - \bar{y} M_{00} = 0 & \mu_{02} &= M_{02} - \bar{y}^2 M_{00} \; . \end{aligned} \tag{5}$$

A general formula for calculating the central moments in terms of the raw moments is

$$\mu_{pq} = \sum_{i=0}^{p+q} \sum_{j=S}^{T} (-1)^{p+q-j} \binom{p}{p-j} \binom{q}{q-i+j} \bar{x}^{\,p-j} \cdot \bar{y}^{\,q-i+j} M_{j,i-j} \tag{6}$$

$$S = \frac{1}{2} \left| (p+i) - |p-i| \right| \qquad T = \frac{1}{2} \left| (i-q) + |i-q| \right|$$

where $\bar{x}$ and $\bar{y}$ are defined in Eq (3) and the notation $\binom{a}{b}$ denotes the usual binomial coefficient. A listing of the central moments up to the fifth order is contained in Appendix A. Perhaps a more convenient form is the recursive formula

$$\mu_{pq} = M_{pq} - \sum_{i=0}^{p+q-1} \sum_{j=S}^{T} \binom{p}{p-j} \binom{q}{q-i+j} \bar{x}^{\,p-j} \cdot \bar{y}^{\,q-i+j} \mu_{j,\,i-j} \tag{7}$$

which calculates μ_{pq} in terms of the raw moment M_{pq} and the lower order central moments. Appendix B lists the recursive central moments for several orders. All moments referred to hereafter will be the central moments unless otherwise stated.

The Algebra of Invariants

The homogeneous polynomial in two variables,

$$f = a_{po}u^p + \binom{p}{1} a_{p-1,1}u^{p-1} v + \cdots + \binom{p}{p-1} a_{1,p-1}uv^{p-1} + a_{op} v^p \tag{8}$$

is a binary quantic of order p. Using the notation common in the study of algebraic invariance, Eq (8) can be written

$$f \equiv (a_{po};\ a_{p-1,\ 1}\ ; \cdots ;\ a_{1,p-1};\ a_{op})(u,\ v)^p. \tag{9}$$

If the general linear transformation,

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} \tag{10}$$

where the determinant Δ is not zero, is applied to f, a new function f' is obtained

$$f' = (a'_{po}\ ;\ a'_{p-1,\ 1}\ ; \cdots ; a'_{1,\ p-1};\ a'_{op})(u',\ v')^p \tag{11}$$

with the same form as f, but with new coefficients. Then, if a homogeneous polynomial I(a) of the coefficients of f exists, such that

$$I(a'_{po}\ ; \cdots ;\ a'_{op}) = \Delta^w\ I(a_{po}; \cdots ;\ a_{op})\ , \tag{12}$$

then I(a) is an algebraic invariant of weight w. For $w \neq 0$, the invariant is called a relative invariant and depends on the transformation. If w=0, then I(a) is an absolute invariant. By eliminating Δ between two relative invariants, a nonintegral absolute invariant can always be obtained.

Moment Invariants

From the Fundamental Theorem of Moment Invariants derived by Hu (Ref 1:181), the algebraic invariants are related to the moments of the same order. That is, if an algebraic form of order p has an invariant of the form of Eq (12), then the moments of order p have the same invariant

$$I(\mu'_{po}; \cdots ; \mu'_{op}) = |J| \quad \Delta^{w} \, I(\mu_{po}; \cdots ; \mu_{op}) \tag{13}$$

but with the additional factor |J|, where J is the Jacobian of the transformation.

By direct substitution of the similitude (change of size) transformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{14}$$

into the definition of moments, every moment is seen to be an invariant,

$$\mu'_{pq} = \alpha^{p+q+2} \quad \mu_{pq}. \tag{15}$$

From the zeroth order relation,

$$\alpha^2 = \frac{\mu_{00}'}{\mu_{00}} \tag{16}$$

and substituting into Eq (15) for α yields the absolute invariants

$$I_{pq} = \frac{\mu_{pq}'}{{\mu_{00}'}^{(p+q+2)/2}} = \frac{\mu_{pq}}{\mu_{00}^{(p+q+2)/2}} . \tag{17}$$

Under the following rotation transformation,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{18}$$

where θ is the angle of rotation, the absolute value of the Jacobian is seen to be $|J| = 1$. Therefore, according to the Fundamental Theorem of Moment Invariants, the moment invariants are the same as the algebraic invariants. If the moments are used as the coefficients of an algebraic quantic,

$$f = (\mu_{p0} ; \cdots ; \mu_{0p})(u, v)^p , \tag{19}$$

then there is a clever technique to derive the necessary invariants. From the transformations,

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} \tag{20}$$

$$\begin{bmatrix} U \\ V \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \qquad \begin{bmatrix} U' \\ V' \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix}, \tag{21}$$

the orthogonal transformation becomes the simple relations

$$U' = Ue^{-i\theta} \quad V' = Ve^{i\theta} \quad . \tag{22}$$

Substitution into Eq (19) yields the relative invariants

$$\begin{aligned} I'_{p0} &= e^{ip\theta} I_{p0} \\ I'_{p-1,1} &= e^{i(p-2)\theta}\ I_{p-1,1} \\ &\vdots \\ I'_{1,p-1} &= e^{-i(p-2)\theta}\ I_{1,p-1} \\ I'_{op} &= e^{-ip\theta}\ I_{0p} \end{aligned} \tag{23}$$

where I and I' denote the corresponding coefficients after the substitutions.

By eliminating the factor $e^{i\theta}$, a complete system of absolute invariants is obtained. For the second and third order moments, the invariants are

$$I_{11} \qquad I_{02}I_{20} \tag{24}$$

$$I_{30}I_{03} \qquad I_{21}I_{12} \qquad I_{30}I_{12}^{3} + I_{03}I_{21}^{3} \tag{25}$$

$$I_{20}I_{12}^{2} + I_{02}I_{21}^{2} \tag{26}$$

$$\frac{1}{i}\,(I_{30}I_{12}^{3} - I_{03}I_{21}^{3}) \;. \tag{27}$$

The last relation changes sign under improper rotation and is called a <u>skew invariant</u>. *Eq (27) is useful for distinguishing* "mirror images". Eqs (24) - (27) can be expressed in terms of the central moments,

$$\mu_{20} + \mu_{02} \tag{28}$$

$$(\mu_{20}-\mu_{02})^{2} + 4\mu_{11}^{2} \tag{29}$$

$$(\mu_{30} - 3\mu_{12})^{2} + (3\mu_{21} - \mu_{03})^{2} \tag{30}$$

$$(\mu_{30} + \mu_{12})^{2} + (\mu_{21} + \mu_{03})^{2} \tag{31}$$

$$(\mu_{30}-3\mu_{12})(\mu_{30}+\mu_{12})\left|(\mu_{30}+\mu_{12})^2-3(\mu_{21}+\mu_{03})^2\right|$$

$$+(3\mu_{21}-\mu_{03})(\mu_{21}+\mu_{03})\left|3(\mu_{30}+\mu_{12})^2-(\mu_{21}+\mu_{03})^2\right| \quad (32)$$

$$(\mu_{20}-\mu_{02})\left|(\mu_{30}+\mu_{12})^2-(\mu_{21}+\mu_{03})^2\right|$$

$$+4\mu_{11}(\mu_{30}+\mu_{12})(\mu_{21}+\mu_{03}) \quad (33)$$

$$(3\mu_{21}-\mu_{03})(\mu_{30}+\mu_{12})\left|(\mu_{30}+\mu_{12})^2-3(\mu_{21}+\mu_{03})^2\right|$$

$$-(\mu_{30}-3\mu_{12})(\mu_{21}+\mu_{03})\left|3(\mu_{30}+\mu_{12})^2-(\mu_{21}+\mu_{03})^2\right| \quad (34)$$

IV. A New Set of Moment Invariants Based on the Methods of Group Theory

A new set of moment invariants is derived from the mathematical methods of the theory of groups. Group theory has been applied to problems in physics where the concepts of symmetry and invariance are important. Such physical notions as parity, spinor, and angular momentum are aspects of group properties. Most of the previous work on moment invariants has been based on the results obtained by Hu (Ref 1: 179-187), which relies on the theory of algebraic invariants. The application of group methods to the same problem represents a new and different viewpoint.

This chapter develops the basic mathematical concepts and definitions leading up to the derivation of a set of invariants. The approach taken is drawn from Hamermesh (Ref 7: 6-30, 77-113). The concepts are then applied to the particular case of invariance with respect to rotation in a plane.

Basic Concepts and Definitions

An abstract group G is defined as a set of elements for which a law of "multiplication" or combination is given such that the "product" ab of any two elements, a and b in G, is defined and also satisfies:

1. If a and b are elements of G, ab is also.

2. Multiplication is associative, $a(bc) = (ab)c$.

3. A set identity element e exists such that $ae = ea = a$ for any element a of the set.

4. For any element a of G, an inverse element $b = a^{-1}$ is also contained in the set such that $ab = ba = e$.

In this context, "multiplication" does not mean an algebraic product, but refers to a rule under which two set elements are combined. If in addition to the above conditions, all elements commute among themselves, the set is <u>abelian</u>. From the concept of abstract groups, a <u>transformation group</u> may be visualized by associating transformations of a set of points as the group elements. This gives a pictorial viewpoint of the group notion.

The number of elements in a group is called the order of the group. The notation a^n implies the product of n elements each equal to a. Negative powers of element a are also defined, $a^{-m} = (a^{-1})^m = (a^m)^{-1}$.

Two groups, G and G', are <u>isomorphic</u> if their elements can be put into one-to-one correspondence which is preserved under combination. If two groups are isomorphic, they have the same structure. Their symbols may differ, but their respective abstract groups are the same. Similarly, a <u>homomorphic</u> mapping of G on G' preserves products, but now several elements of G may have the same image in G'.

<u>Group Representations</u>

In a vector space L, a set of operators forms a group

if it satisfies the definition given above. The product of two operators A and B on the vector $\overline{X}$ means the single operator $C\overline{X} = A(B\overline{X})$ for all $\overline{X}$ in L. The identity operator leaves the operand unchanged, and all operators have an inverse. If the vector space L is mapped onto a second space L', via an operator S, an isomorphic group of operators is obtained in L' which are transforms of the vectors in L, $A' = SAS^{-1}$.

If a group G is mapped homomorphically onto a group of operators D(G) in the vector space L, the operator group D(G) is a <u>representation</u> of G in the <u>representation space</u> L. If L has dimension n, the representation has <u>degree</u> n or is an <u>n-dimensional</u> representation. If the homomorphism reduces to an isomorphic mapping, the representation D(G) is <u>faithful</u>, and the order of D(G) is equal to the order g of G. The operator corresponding to an element R in group G will be denoted by D(R). If R and S are elements of G, then

$$D(RS) = D(R)D(S) \tag{35}$$

$$D(R^{-1}) = \left\{D(R)\right\}^{-1} \tag{36}$$

$$D(E) = 1. \tag{37}$$

where E represents the identity transformation.

A <u>linear representation</u> is a group representation in terms of linear operators. All representations hereafter are assumed to be linear, unless specifically noted otherwise.

Given a basis in L, the linear operators of the representation can be represented by their matrix representatives. Group G is then mapped homomorphically onto a group of matrices $\underline{D}(G)$, giving a <u>matrix representation</u> of the group. The matrices are nonsingular and

$$D_{ij}(E) = \delta_{ij}$$

$$D_{ij}(RS) = \sum_{K} D_{ik}(R)\ D_{kj}(S) \equiv D_{ik}(R)\ D_{kj}(S). \tag{38}$$

If the basis in the n-dimensional space L is changed, the matrices $\underline{D}(R)$ will be transformed by some matrix $\underline{C}$,

$$\underline{D}'(R) = \underline{C}\ \underline{D}(R)\ \underline{C}^{-1}. \tag{39}$$

The transformed matrix also gives a representation of the group and is <u>equivalent</u> to $\underline{D}(R)$. Equivalent representations have the same structure, even though the matrices appear dissimilar.

An intrinsic property of a representation $\underline{D}(R)$, independent of basis, is the <u>trace</u> $\chi(R)$, or the sum of the

diagonal elements of the matrix representation.

$$\chi(R) = \sum_i D_{ii}(R) \tag{40}$$

The trace is called the <u>character</u> of R in the representation D. Equivalent representations have the same set of characters.

Given a transformation T belonging to a transformation group G (or to the group associated with the matrix representation $\underline{D}(G)$), new representations can be constructed. Transformation T acts on $\overline{X}$ to produce $\overline{X}'$: $\overline{X}' = T\overline{X}$. A linear operator O_T associated with T acts on some function $\Psi(\overline{X})$ such that

$$\Psi'(\overline{X}') \equiv O_T\Psi(\overline{X}') = \Psi(\overline{X}), \tag{41}$$

if $\overline{X}' = T\overline{X}$. In other words, the transformed function Ψ' takes the same value at the image point $\overline{X}'$ as the original function Ψ at the object point $\overline{X}$. Another way of putting it is that the effect of the operator O_T is as the point P is transformed to image point P', it carries with it the value of Ψ at P.

Therefore

$$O_T\Psi(T\overline{X}) = \Psi(\overline{X}) \tag{42}$$

or

$$O_T\Psi(\overline{X}) = \Psi(T^{-1}\overline{X}) \ . \tag{43}$$

The general procedure for constructing representations should now be clear. Apply all the operators O_R corresponding to the transformation group to each of any set of linearly independent functions. A set of functions is obtained which can be expressed linearly in terms of n of them, $\Psi_1, \Psi_2, \ldots, \Psi_n$. Applying any operator O_R to these functions results in a linear combination of the same n functions,

$$O_R \Psi_j = \sum_{i=1}^{n} \Psi_i \quad D_{ij}(R) \quad j=1,\ldots,n. \tag{44}$$

The correspondent of the element R in the representation is then the matrix $\underline{D}(R)$.

Reducible and Irreducible Representations

Given a representation D, it is possible to describe it in terms of "simpler" representations. Roughly, "simpler" means that the representations have the lowest dimensions possible. In general, if a basis exists in which all matrices $\underline{D}(R)$ of an n-dimensional representation can be brought to the form

$$\underline{D}(R) = \left[\begin{array}{c:c} \underline{D}^{(1)}(R) & \underline{A}(R) \\ \hdashline \underline{O} & \underline{D}^{(2)}(R) \end{array}\right] \tag{45}$$

where $\underline{D}^{(1)}(R)$ has m × m dimensions, $\underline{D}^{(2)}(R)$ is (n-m) × (n-m),

and $\underline{A}(R)$ is m x (n-m), then the representation is <u>reducible</u>. An intrinsic indicator of reducibility is that there exist some subspace of dimension less than the representation which is invariant under the transformations of the group. If it is possible to find a basis such that the representation matrices have the form of Eq (45) with $\underline{A}(R)$ being the zero matrix, the representation is <u>fully reducible</u>. A representation for which there is no invariant proper subspace is <u>irreducible</u>.

Among the irreducible representations there may be several which are equivalent. These, of course, must have the same dimensionality. Equivalent irreducible representations are not distinct, and the same symbol can be used for them. Also, a representation of a group may contain a particular irreducible representation several times.

<u>Invariance of Functions</u>

Recalling Eq (43), $O_R\Psi(\bar{X}) = \Psi(R^{-1}\bar{X})$, it is clear that O_R operating on Ψ replaces $\bar{X}$ by $R^{-1}\bar{X}$. It is possible that $O_R\Psi$ is identical with Ψ,

$$O_R\Psi(\bar{X}) \equiv \Psi(\bar{X}) \tag{46}$$

so that

$$\begin{aligned} \Psi(\bar{X}) &= \Psi(R^{-1}\bar{X}) \\ \Psi(R\bar{X}) &= \Psi(\bar{X}) \ . \end{aligned} \tag{47}$$

In this case, the function Ψ takes on the same value at the image point as the object point, and is <u>invariant</u> under the operator O_R, or more briefly, under the transformation R. To test for invariance of a function, the arguments are replaced by their images to see if the same expression is obtained.

In the theory of representations, a complex number $(\bar{X}, \bar{Y})$, called the <u>scalar product</u>, is associated with each pair of vectors $\bar{X}$ and $\bar{Y}$ in vector space L such that:

$$1. \quad (\bar{X}, \bar{Y}) = (\bar{Y}, \bar{X})^* \tag{48}$$

$$2. \quad (\bar{X}, \alpha\bar{Y}) = \alpha(\bar{X}, \bar{Y}) \tag{49}$$

$$3. \quad (\bar{X}_1 + \bar{X}_2, \bar{Y}) = (\bar{X}_1, \bar{Y}) + (\bar{X}_2, \bar{Y}) \tag{50}$$

$$4. \quad (\bar{X}, \bar{X}) \geq 0 \tag{51}$$

and $(\bar{X}, \bar{X}) = 0$ only if $\bar{X}$ is the zero vector. A space in which a scalar product is defined is called a <u>unitary space</u>. In defining the scalar product, a basis was not mentioned, which means that $(\bar{X}, \bar{Y})$ is an intrinsic property of the vectors $\bar{X}$ and $\bar{Y}$, and is independent of basis.

Any function satisfying Eqs (48) - (51) can be used to define a scalar product in the space L. One definition of $(\bar{X}, \bar{Y})$ is to write it as a function of the vector coordinates x_i and y_i in a particular basis. If the basis vectors are $\bar{u}_i$, a <u>metric matrix</u> M is defined by

$$m_{ij} = (\bar{u}_i, \bar{u}_j) \quad . \tag{52}$$

From Eq (48), it is clear that

$$m_{ij} = m^*_{ji} \tag{53}$$

$$\underline{M} = M^{\dagger} \tag{54}$$

$$(\underline{M}^{\dagger})_{ij} = m^*_{ji} \tag{55}$$

Where $\underline{M}^{\dagger}$ is the conjugate of matrix $\underline{M}$ transposed and is the <u>adjoint</u> of $\underline{M}$. A matrix which is identical to its adjoint is called <u>self-adjoint</u> or <u>Hermitian.</u> Therefore, from the above, $\underline{M}$ must be Hermitian. If vectors $\bar{X}$ and $\bar{Y}$ are expanded in the basis $\bar{u}_i$,

$$\bar{X} = x_1\bar{u}_1 + \cdots + x_n\bar{u}_n \equiv x_i\bar{u}_i \tag{56}$$

$$\bar{Y} = y_1\bar{u}_1 + \cdots + y_n\bar{u}_n \equiv y_j\bar{u}_j \tag{57}$$

then it is clear that

$$\begin{aligned} (\bar{X},\bar{Y}) &= (x_i\bar{u}_i\ ,\ \ y_j\bar{u}_j) \\ (X,Y) &= \bar{X}^{\dagger}\underline{M}\ \bar{Y} \end{aligned} \tag{58}$$

where $\bar{X}$ and $\bar{Y}$ are column matrices and $\bar{X}$ is the adjoint of $\bar{X}$.

Two vectors in a unitary space are <u>orthogonal</u> if their scalar product is zero, $(\bar{X}, \bar{Y}) = 0$. Given a basis $\bar{v}_i$ in the unitary space L, a new set of basis vectors $\bar{u}_i$ that are mutually orthogonal can always be constructed and forms an <u>orthonormal basis</u> if $(\bar{u}_i, \bar{u}_j) = \delta_{ij}$.

In a unitary space L, the <u>distance</u> $|\bar{X} - \bar{Y}|$ between two vectors is defined from

$$|\bar{X} - \bar{Y}|^2 = (\bar{X} - \bar{Y}, \bar{X} - \bar{Y}). \tag{59}$$

A sequence of vectors $\bar{X}_n$ $(n = 1, \ldots, \infty)$ in L is said to <u>converge</u> to $\bar{X}$ in L if $\lim_{n \to \infty} |\bar{X}_n - \bar{X}| = 0$. The sequence of vectors $\bar{X}_n$ is said to be a <u>fundamental sequence</u>, if $\lim_{n \to \infty} |\bar{X}_m - \bar{X}_n| = 0$. If every fundamental sequence converges to a vector in the vector space L, the space is <u>complete</u>. A <u>complete unitary</u> space is called a <u>Hilbert space</u>. The unitary spaces of finite dimension are complete. Infinite dimensional representations will be restricted to representations by linear operators in a Hilbert space under the condition that the operators are continuous.

To tie the preceding paragraphs together with invariants, assume that an irreducible representation D(R) is unitary. Then the scalar product of vectors in the Hilbert space of the representation is invariant.

The inverse transpose of each of the matrices of an irreducible representation is also a representation of the group and is called the <u>adjoint representation</u> $\bar{D}$ (R),

$$\bar{D}(R) \equiv \tilde{D}^{-1}(R). \tag{60}$$

Likewise, the complex conjugate of D(R) is the <u>complex conjugate representation</u> $D^*(R)$. If the adjoint representation and the complex conjugate representation are equivalent

$$\tilde{D}^{-1}(R) \approx D^*(R), \tag{61}$$

there exists a matrix $\underline{F}$ such that

$$\begin{aligned} \underline{D}^*(R) &= \tilde{\underline{F}}^{-1}\,\tilde{\underline{D}}^{-1}(R)\tilde{\underline{F}} \\ \underline{D}^\dagger(R) &= \underline{F}\underline{D}^{-1}(R)\underline{F}^{-1} \end{aligned} \tag{62}$$

or

$$\underline{D}^\dagger(R)\underline{F}\underline{D}(R) = \underline{F}. \tag{63}$$

Then $(\bar{Y}, \underline{F}\,\bar{X})$ is invariant under all transformations of the group G. In fact, for an irreducible representation, <u>there can be no more than one invariant of this form</u>.

Expansion of Function in Terms of an Irreducible Representation Basis

As noted previously, a representation of group G can be constructed by applying the transformations of G to any function Ψ. Then Ψ will be a base function or a linear combination of the basis. This can be extended further by decomposing the representation into its irreducible constituents. In other words, any function Ψ can be expressed as a sum of the base functions in the irreducible representations,

$$\Psi = \sum_{\nu} \sum_{i=1}^{n_\nu} \Psi_i^{(\nu)}. \tag{64}$$

The base functions for the νth irreducible unitary representation satisfies

$$O_R \Psi_i^{(\nu)} = \sum_j \Psi_j^{(\nu)} D_{ji}^{(\nu)}(R). \tag{65}$$

The desired aimpoint is to find the $\Psi_i^{(\nu)}$ given the function Ψ. In other words, how can the given function be resolved into a sum of functions, each belonging to a particular row of an irreducible representation?

If Eq (65) is multiplied by $D_{lm}^{(\mu)*}(R)$ and summed over the entire group

$$\sum_R D_{lm}^{(\mu)*}(R)\; O_R\; \Psi_i^{(\nu)} = \sum_j \Psi_j \sum_R D_{lm}^{(\mu)*}(R)\; D_{ji}^{(\nu)}(R)$$

$$= \frac{g}{\eta_\nu}\, \Psi_l^{(\nu)}\; \delta_{mi}\; \delta_{\mu\nu} \tag{66}$$

due to the orthogonality relations of Appendix D. For the case $m = k = i$ and $\mu = \nu$

$$\Psi_l^{(\nu)} = \frac{\eta_\nu}{g} \sum_R D_{lk}^{(\nu)*}(R)\; O_R\; \Psi_k^{(\nu)}. \tag{67}$$

That this is a necessary and sufficient condition on the $\Psi_i^{(\nu)}$ such that Eq (65) is satisfied is proven in Appendix E.

A projection operator is defined from Eq (66) as

$$P_i^{(\mu)} = \frac{\eta_\mu}{g} \sum_R D_{ii}^{(\mu)*}(R)\; O_R \tag{68}$$

such that

$$P_i^{(\mu)}\; \Psi_j^{(\nu)} = \Psi_i^{(\mu)}\; \delta_{\mu\nu}\delta_{ij}. \tag{69}$$

Appendix E shows that if the $P_i^{(\mu)}$ is applied to Eq (64), the result

$$\Psi_i^{(\mu)} = \frac{\eta_\mu}{g} \sum_R D_{ii}^{(\mu)*}(R)\; O_R \tag{70}$$

is obtained or,

$$\Psi_i^{(\mu)} = P_i^{(\mu)}\Psi , \tag{71}$$

which is the desired end. Briefly, to find the projection of a function onto a basis function of an irreducible representation, multiply by the basis function and sum over the entire group. Another way of viewing it is that the projection operator finds the component of Ψ in the "direction" of the basis function.

Application of Group Theory to Moment Invariants

The basic tools are now available to derive a set of moment invariants under the rotation transformation. The rotation transformation of Eq (18)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \mathrm{COS}\theta & \mathrm{SIN}\theta \\ -\mathrm{SIN}\theta & \mathrm{COS}\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{18}$$

forms an infinite continuous, abelian transformation group over the region $\theta \; \varepsilon \left| 0, 2\pi \right|$. For simplicity of notation, $\mathrm{COS}m\theta \equiv C_m$ and $\mathrm{SIN}m\theta \equiv S_m$. The approach taken is simple: (1) Form a reducible matrix representation of the rotation group, (2) use the projection operator to form vectors in terms of the irreducible basis functions, and (3) form invariants of the scalar product form. As before, all reference to moments is taken to mean central moments.

Using the notation of the expected value operator, the pqth moment can be written

$$\mu_{pq} = E\left\{x^p y^q\right\}. \tag{72}$$

Under the rotation operation, the new moments become

$$\mu'_{pq} = E\left\{(xC+yS)^p(-xS+yC)^q\right\}. \tag{73}$$

The zeroth order moment remains the same,

$$\mu'_{00} = \mu_{00}, \tag{74}$$

and the first order moments in terms of the unrotated moments become

$$\begin{bmatrix} \mu'_{01} \\ \mu'_{10} \end{bmatrix} = \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{bmatrix} \mu_{01} \\ \mu_{10} \end{bmatrix} . \tag{75}$$

Thus, the reducible representation is obtained. Appendix F lists the rotated moments up to the sixth order. In order to obtain the same likeness as in Appendix F, the trigonometric identities of Appendix G will be useful.

The projection operator, previously defined in Eq (68) can now be used to expand the transformed moments above in terms of the basis of the associated irreducible represent-

ation. By applying Eq (70) to the reducible representations above, a vector in a Hilbert space is obtained whose components are

$$X_{pq}^{(m)} = \frac{\eta_m}{2\pi} \int_0^{2\pi} C_m \cdot \mu_{pq}' \, d\theta$$

$$Y_{pq}^{(m)} = \frac{\eta_m}{2\pi} \int_0^{2\pi} S_m \cdot \mu_{pq}' \, d\theta \tag{76}$$

where an integration instead of a summation is performed since the group is continuous. The irreducible representation basis is composed of terms of the form C_m and S_m ($m = 0, 1, \ldots$) where m can be thought of as a "frequency" of rotation. The effect of the projection operator is to "peel out" that component of the pqth representation which has frequency m. Thus for the zeroth order,

$$X_{00}^{(0)} = 2\mu_{00}$$

$$Y_{00}^{(0)} = 0, \tag{77}$$

and for the first order,

$$X_{01}^{(1)} = \mu_{01}$$

$$Y_{01}^{(1)} = \mu_{10} \, . \tag{78}$$

It is apparent that if a particular representation does not contain any terms with a specified frequency m, then the projection operator gives the zero vector. This might be expected, since the term does not have any components associated with the specified basis vector. The projected moment vectors up to the sixth order are listed in Appendix H. It may be observed that a distinct vector is not formed for every moment within an order. For instance, the second order vectors from Appendix H

$$X_{02}^{(0)} = \mu_{20} + \mu_{02} \qquad Y_{02}^{(0)} = 0 \qquad \text{(H-3)}$$

$$X_{02}^{(2)} = -\frac{1}{2}(\mu_{20} - \mu_{02}) \qquad Y_{02}^{(2)} = \mu_{11} \qquad \text{(H-4)}$$

do not contain vectors corresponding to μ'_{11} and μ'_{20}. However, if the projection operator is applied to these cases, the result

$$X_{11}^{(2)} = \frac{1}{2}(\mu_{02} - \mu_{20}) \qquad Y_{11}^{(2)} = \mu_{11} \qquad (79)$$

$$X_{20}^{(0)} = \mu_{02} + \mu_{20} \qquad Y_{20}^{(0)} = 0 \qquad (80)$$

$$X_{20}^{(2)} = \frac{1}{2}(\mu_{02} - \mu_{20}) \qquad Y_{20}^{(2)} = \mu_{11} \qquad (81)$$

is obtained. Comparing Eqs (79) - (81) to Eqs (H-3) - (H-4), it is seen that Eqs (79) and (81) are the negative of Eq (H-4),

and Eq (80) is the same as Eq (H-3). This is expected since as was previously noted, among the irreducible representations there may be several which are equivalent and are not distinguishable.

In a previous section, it was stated that the only form of invariants is a scalar product ($\bar{Y}$, $\underline{F}$ $\bar{X}$) between vectors within an irreducible representation. Since the rotation group forms a unitary representation, the matrix $\underline{F}$ must be a multiple of the identity matrix by Schur's Lemma (Appendix D). Therefore, the invariants are of the form ($\bar{Y}$, $\bar{X}$), a direct scalar product. For the vectors derived above for the rotation group, the invariants are formed only between vectors of the same frequency, or more fundamentally, between vectors from the same representation. It is evident that invariants can be formed between vectors of differing moment orders, but with the same frequency, since they come from equivalent representations. For example, an invariant is formed within the second order moments,

$$
\begin{aligned}
{}_{02}I_{02}^{(0)} &= X_{02}^{(0)}\ X_{02}^{(0)} + Y_{02}^{(0)}\ Y_{0}^{(2)} \\
&= (\mu_{20} + \mu_{02})^2\ , \qquad (82)
\end{aligned}
$$

by forming the scalar product of the zero frequency vector corresponding to μ_{02} with itself. Another invariant is

formed between second and fourth order moments,

$$ {}_{02}I_{04}^{(0)} = \frac{3}{4}\,(\mu_{20}+\mu_{02})\,(\mu_{40}+2\mu_{22}+\mu_{04}). \tag{83} $$

Appendix I lists the invariants formed from the vectors of Appendix H.

A Complete Set of Moment Invariants

A "complete" set of invariants in the sense to be considered means that the moments from which the invariants were formed can be found, and in turn, the image itself can be reconstructed. However, this is not completely true. For a particular image, a unique and complete set of two-dimensional moments of all orders exists. A specific and unique relationship exists between each and every moment. To obtain independence with respect to translation, the central moments were introduced. However, invoking the central moments discards the location of the image centroid. Independence with respect to rotation is obtained at the expense of the angle of rotation of the image. But a unique relationship still exists between the remaining moments, and in fact, the invariants are a description of that relationship. Thus, to obtain invariance with respect to a transformation, some information about the image is discarded.

From Appendix H, the vectors from the second and third order moments are

$$X_{02}^{0} = \mu_{20}+\mu_{02} \qquad Y_{02}^{0} = 0 \tag{H-3}$$

$$X_{02}^{2} = -\frac{1}{2} (\mu_{20} - \mu_{02}) \qquad Y_{02}^{2} = -\mu_{11} \tag{H-4}$$

$$X_{03}^{1} = \frac{3}{4} (\mu_{21} + \mu_{03}) \qquad Y_{03}^{1} = -\frac{3}{4}(\mu_{30}+\mu_{12}) \tag{H-5}$$

$$X_{03}^{3} = -\frac{1}{4} (3\mu_{21}-\mu_{03}) \qquad Y_{03}^{3} = \frac{1}{4} (\mu_{30}-3\mu_{12}) \tag{H-6}$$

$$X_{12}^{1} = \frac{1}{4} (\mu_{30} + \mu_{12}) \qquad Y_{12}^{1} = \frac{1}{4} (\mu_{21}+\mu_{03}) \; , \tag{H-7}$$

where the superscripts indicate the frequency of rotation with respect to the image rotation. Since there is not a common frequency between the two orders, it appears that no invariant of the inner product form can be constructed. However, if the reducible matrix representation associated with the second order moments is cubed and the representation for the third order is squared, common frequencies are obtained after the projection operators is applied. In particular, for the second order

$$(\mu_{02}')^{3} = \frac{1}{8} \Big[\left(\frac{1}{4} C_6 + \frac{3}{2} C_4 + \frac{15}{4} C_2 + \frac{5}{2} \right) \mu_{02}^{3}$$

$$-\left(\frac{3}{2} S_6+6S_4 +6S_2\right) \mu_{02}^{2}\mu_{11} - \left(\frac{3}{4} C_6 + \frac{5}{2} C_4 - \frac{3}{4} C_2 - 5_{2}\right)\mu_{02}^{2}\mu_{20}$$

$$-(-2S_6 +6S_2) \mu_{11}^{3} + (-3C_6 -6C_4 +3C_2 +6) \mu_{02}\mu_{11}^{2}$$

$$+(3S_6 - 9S_2)\mu_{02}\mu_{20}\mu_{11}$$

$$+(\frac{3}{4} C_6 - \frac{3}{2} C_4 - \frac{3}{4} C_2 + \frac{3}{2})\ \mu_{20}^2 \mu_{02}$$

$$-(-3C_6 + 6C_4 + 5C_2 - 6)\ \mu_{20}\mu_{11}^2$$

$$-(\frac{3}{2} S_6 - 6S_4 + \frac{15}{2} S_2)\mu_{20}^2 \mu_{11}$$

$$-(\frac{1}{4} C_6 - \frac{3}{2} C_4 + \frac{15}{4} C_2 - \frac{5}{2})\mu_{20}^3\Big] , \qquad (84)$$

and after applying the projection operator Eq (76)

$$X_2^6 = \frac{1}{4} \mu_{02}^3 - \frac{3}{4} \mu_{02}^2 \mu_{20} - 3\mu_{02}\mu_{11}^2 + \frac{3}{4} \mu_{20}^2\mu_{02}$$

$$+3\mu_{20}\mu_{11}^2 - \frac{1}{4} \mu_{20}^3$$

$$Y_2^6 = -\frac{3}{2} \mu_{02}^2 \mu_{11} + 2\mu_{11}^3 + 3\mu_{02}\mu_{20}\mu_{11} - \frac{3}{2} \mu_{20}^2 \mu_{11} . \qquad (85)$$

For the third order

$$(\mu'_{03})^2 = \frac{1}{16} \Big[\frac{1}{2}(C_6 + 6C_4 + 15C_2 + 10)\ \mu_{03}^2 - 3(S_6 + 4S_4 + 5S_2)\mu_{12}\mu_{13}$$

$$-3(C_6 + 2C_4 - C_2 - 2)\mu_{21}\mu_{03} + \frac{9}{2}(-C_6 - C_4 + C_2 + 2)\mu_{12}^2$$

$$+(S_6 -3S_2)\mu_{30}\mu_{03} +9(S_6 -3S_2)\mu_{21}\mu_{12}$$

$$-3(-C_6 +2C_4 +C_2-2)\ \mu_{30}\mu_{12}$$

$$+\frac{9}{2}(C_6 -2C_4 -C_2 +2)\ \mu_{21}^2 +3(S_6 -4S_4 +5S_2)\mu_{30}\mu_{21}$$

$$+\frac{1}{2}(-C_6 +6C_4 -15C_2 +10)\mu_{30}^2\Big] \tag{86}$$

and

$$X_3^{\ 6} = \mu_{03}^2 -6\mu_{21}\mu_{03} +9\mu_{21}^2 -9\mu_{12}^2 +6\mu_{30}\mu_{12} -\mu_{30}^2$$

$$Y_3^{\ 6} = 2(-3\mu_{12}\mu_{03} -\mu_{30}\mu_{03} +9\mu_{21}\mu_{12} +3\mu_{30}\mu_{21}) \tag{87}$$

$$X_3^{\ 2} = 5\mu_{03}^2 +2\mu_{21}\mu_{03} -3\mu_{21}^2 +3\mu_{12}^2 -2\mu_{30}\mu_{12} -5\mu_{30}^2$$

$$Y_3^{\ 2} = 2(-5\mu_{12}\mu_{03} -\mu_{30}\mu_{03} -9\mu_{21}\mu_{12} +5\mu_{30}\mu_{21}). \tag{88}$$

Scalar products between the vectors of the same frequency can now be formed

$${}_3I_2^{(2)} = (\mu_{02}-\mu_{20})(5\mu_{03}^2 +2\mu_{21}\mu_{03} -3\mu_{21}^2 +3\mu_{12}^2 -2\mu_{30}\mu_{12} -5\mu_{30}^2)$$

$$+4\mu_{11}(5\mu_{12}\mu_{03} +\mu_{30}\mu_{03}+9\mu_{21}\mu_{12} -5\mu_{30}\mu_{21}) \tag{89}$$

$$
{}_3I_2^{(6)} = (\mu_{03}^2 - 6\,\mu_{21}\mu_{03} + 9\mu_{21}^2 - 9\mu_{12}^2 + 6\mu_{30}\mu_{12} - \mu_{30}^2)
$$

$$
\cdot(\mu_{02}^3 - 3\mu_{02}^2\mu_{20} - 12\mu_{02}\mu_{11}^2 + 2\mu_{20}\mu_{11}^2 + 3\mu_{20}^2\mu_{02} - \mu_{20}^3)
$$

$$
+4\mu_{11}\ (3\mu_{12}\mu_{03} + \mu_{30}\mu_{03} - 9\mu_{21}\mu_{12} - 3\mu_{30}\mu_{21})
$$

$$
(3\mu_{02}^2 - 4\mu_{11}^2 - 6\mu_{02}\mu_{20} + 3\mu_{20}^2)\ . \tag{90}
$$

Similarly, all orders of moments can be linked to form invariants. A complete set of moment invariants with respect to rotation and translation through the fourth order is listed in Appendix J.

Comparing Appendix J with the moment invariants derived from the algebra of invariants by Hu, Eqs (28) - (34), a striking similarity can be observed. Eqs (28) - (31) are virtually identical with Eqs (J-1) - (J-4), and Eq (32) is very similar to Eq (J-5) except for coefficients. It is also observed that there are (p+1) invariants derived by group theory methods, where p is the moment order. This is exactly the same as in Chapter III. It can be concluded that the two sets of moment invariants, each derived by different methods, group theory and algebra of invariants, are equivalent.

V. Image Signal Effects

It is obvious that the image signal over which the moment invariants are to be calculated depends on many parameters of the optical system, the object, and the transmission channel. The effects of the variation of the parameters must be taken into account in designing a recognition system. The discussion to follow is oriented toward visual data of aircraft in an atmospheric environment.

Invariance With Respect to Optical Path Length

As an object is moved along the optical axis, it is evident that the image signal of an optical system will vary. A first order effect is obviously a change in size. A second order effect will be the appearance or disappearance of small details of the object. This second order effect diminishes as the optical path length from sensor to object increases. For aircraft type problems, this may be negligible.

The change of size problem is mainly a matter of normalization of the image moments. A radius of gyration is defined to be

$$r = \sqrt{\mu_{20} + \mu_{02}} \tag{91}$$

and is directly proportional to the image size or inversely proportional to the distance of the object along the optical path length. The product of the radius of gyration of the image and the range of the object is a constant. Therefore, the general moment may be normalized to be

$$\mu''_{pq} = \mu_{pq} / (\mu_{20} + \mu_{02})^{(p+q+2)/4} \tag{92}$$

A second choice of normalization follows the rule derived earlier, Eq (17), under the similitude transformation.

$$I = \mu_{pq}/\mu_{00}^{(p+q+2)/2}, \quad p+q = 2,3,\cdots \tag{17}$$

This corresponds to having the average scene brightness μ_{00} always equal to unity. Therefore, by dividing the central moments by μ_{00}

$$\mu''_{pq} = \mu_{pq} / \mu_{00} \qquad p+q = 0,1,2 \cdots \tag{93}$$

the size change effect is eliminated. Of course, such normalization is not unique. An advantage of this method is that invariance with respect to scaling is obtained. That is, the scaled moment becomes

$$\begin{aligned} \mu'_{pq} &= \iint x^p y^q \left|\alpha \rho(x,y)\right| \, dxdy \\ &= \alpha \mu_{pq} \end{aligned} \tag{94}$$

and after normalization

$$\begin{aligned} \mu''_{pq} &= \alpha\mu_{pq} / \alpha\mu_{00} \\ &= \mu_{pq} / \mu_{00} \end{aligned} \tag{95}$$

which is the desired result.

Changes in illumination

Another problem that occurs is that under different conditions of illumination, the image moments will vary. For example, a video camera takes a frame of an aircraft at sunrise and at sunset at the same range and orientation. The shadows on the aircraft will be different, causing a different weight to be placed on a particular area. Dudani eliminated this problem by using only the boundary and silhouette of the target (Ref 4: 33-34). However, it is obvious that some information is not being used in this case.

If instead of a video detector, an IR (infrared) sensor system is used, the problem of scene illumination is eliminated, or at least reduced. IR signatures are presently being studied for identification purposes. Variation in illumination may also be incorporated into a statistical classifier.

Aircraft/Missile Engine Plume

For the particular class of targets which include flying aircraft/missiles, there is the unwanted feature of the engine plume.

It will be assumed that the entire aircraft is included in the field of view; otherwise, it cannot easily be identified. Depending on the range from the optical system to the target and upon the location of the aircraft within the sensor field of view, the engine plume may be totally, partially, or not at all, within the image. Also, the plume might tend to dominate the imagery data. Therefore, the identification problem becomes much more difficult and it is desirable to "gate off" the engine plume.

In general, a distinctive feature of the plume is required to indicate the dividing point of the aircraft and the plume. One such feature may be that the plume is hottest at the exit of the engine. By scanning along the axis of the plume, it may be possible to pick out this point and incorporate its coordinates to gate off the unwanted portion of the image. Frequency discrimination may also be employed as a distinguishing feature.

Target Background Clutter

As in the case of the engine plume, the target background represents extraneous and unwanted information. Also, in order at least to approximate the finiteness condition of the uniqueness property of moments, a background of zero intensity is ideally desired. For the case of aircraft data being considered, this unwanted information may take the form of blue sky or clouds.

Assuming an approximately constant intensity background, a rough solution is to subtract from the image, point by point, a constant intensity depending on some parameters of the image. If the intensity at any point becomes negative, it is set to zero. Presumably, the background clutter would be eliminated or reduced.

As a first guess, the threshold constant was taken to be the difference between the minimum intensity and a constant times the difference between the maximum and minimum intensity image,

$$\text{THRSHD} = \text{MIN}+\text{CONSTANT} \cdot (\text{MAX}-\text{MIN}) \ . \qquad (33)$$

The above procedure was applied to imagery data of a Hawk missile in flight and the moment invariants were calculated for the resulting image. Figures (1) - (7) show the variation of the seven moment invariants derived earlier, Eqs (28-34) as a function of the threshold constant. The increment between each plotted point is $\frac{1}{100} \cdot$ (MAX-MIN). A jump in the value of the moment invariants is observed at the second point and a gradual variation over the succeeding points. This indicates that for a threshold of

$$\text{THRSHD} = \text{MIN} + \frac{2}{100} \ (\text{MAX-MIN}) \qquad (34)$$

the background effect has been minimized and the image of the Hawk missile is the dominant factor.

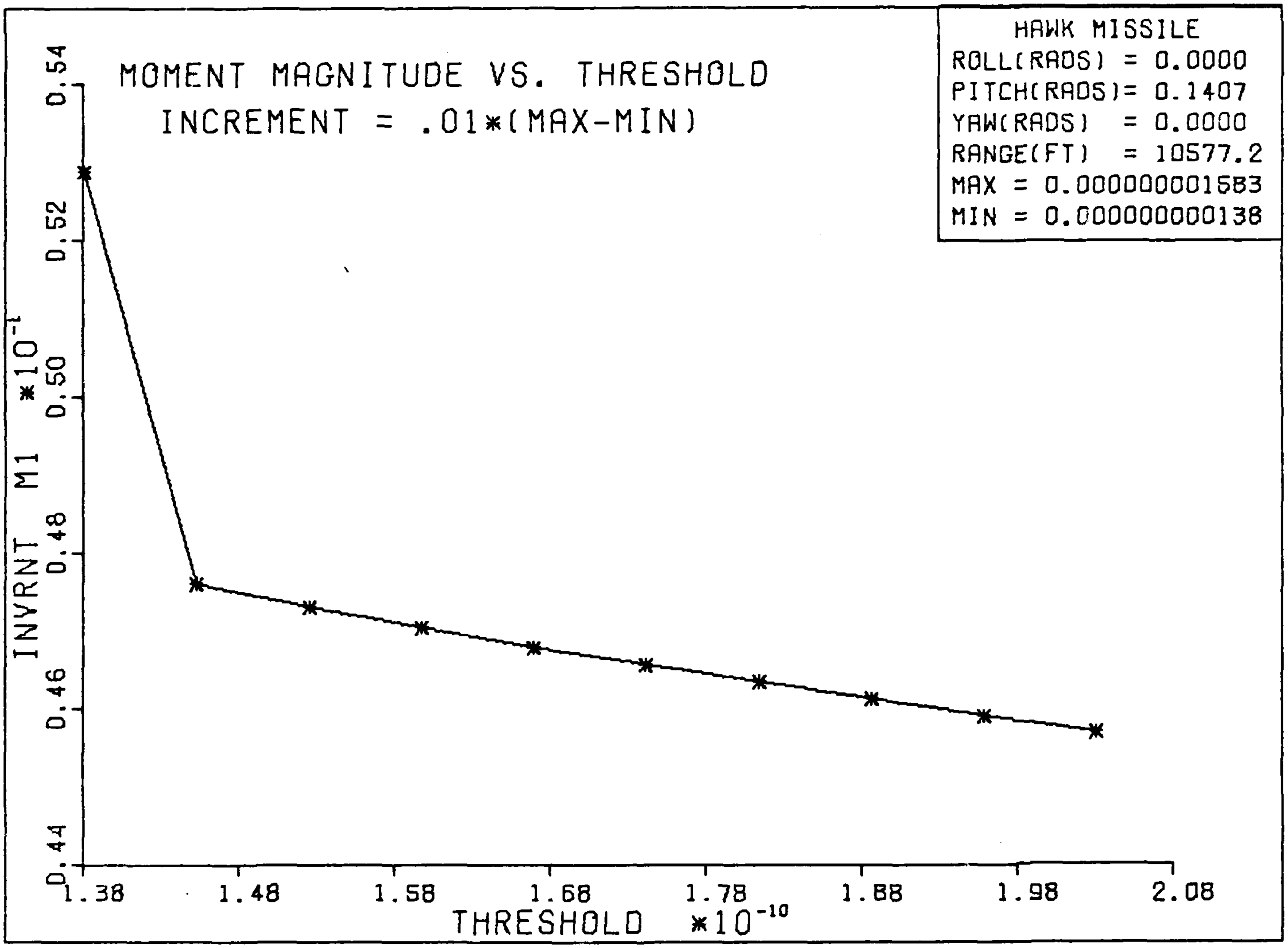

Figure 1. Variation of Moment Invariants versus Threshold.

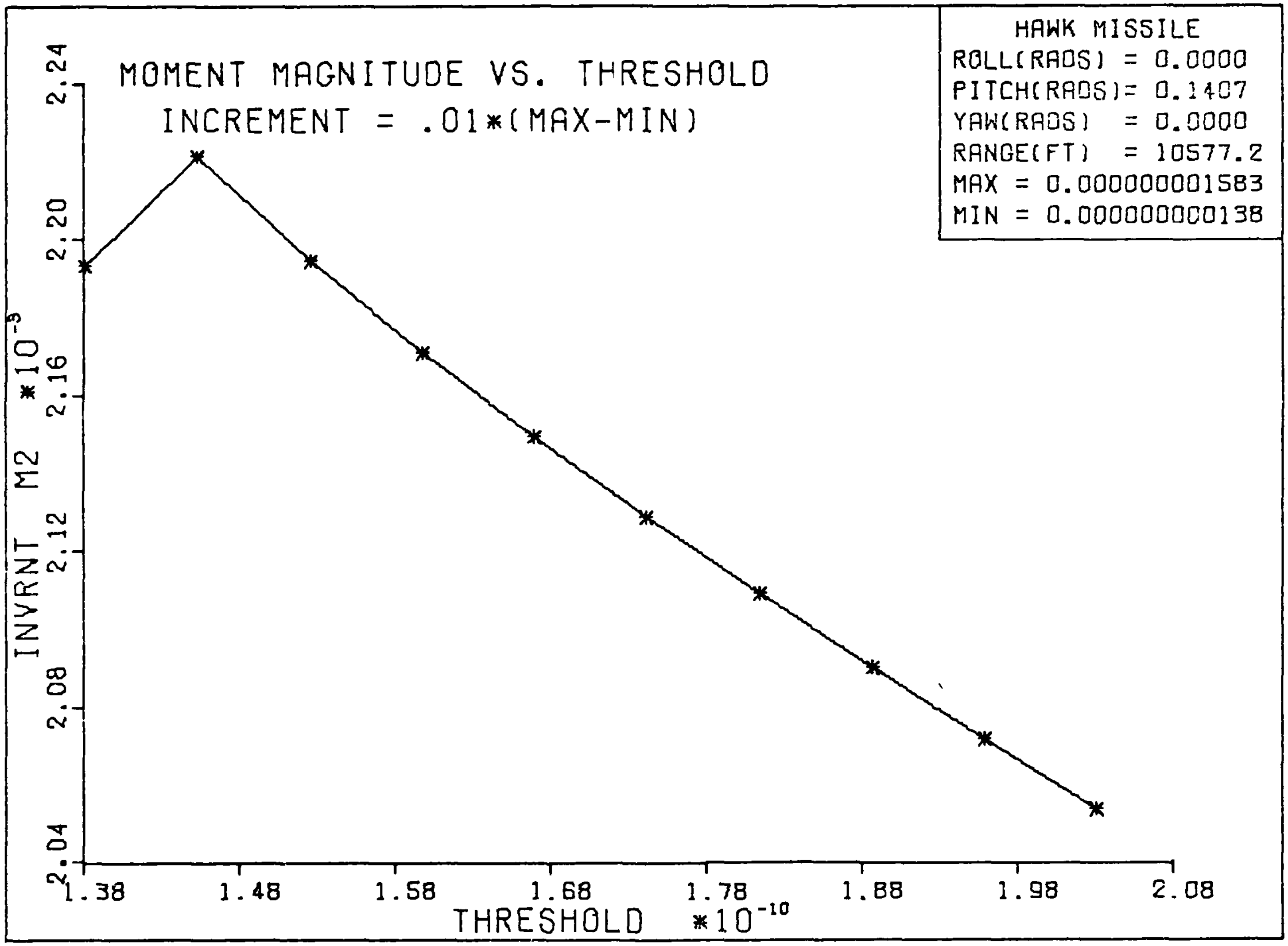

Figure 2. Variation of Moment Invariants versus Threshold.

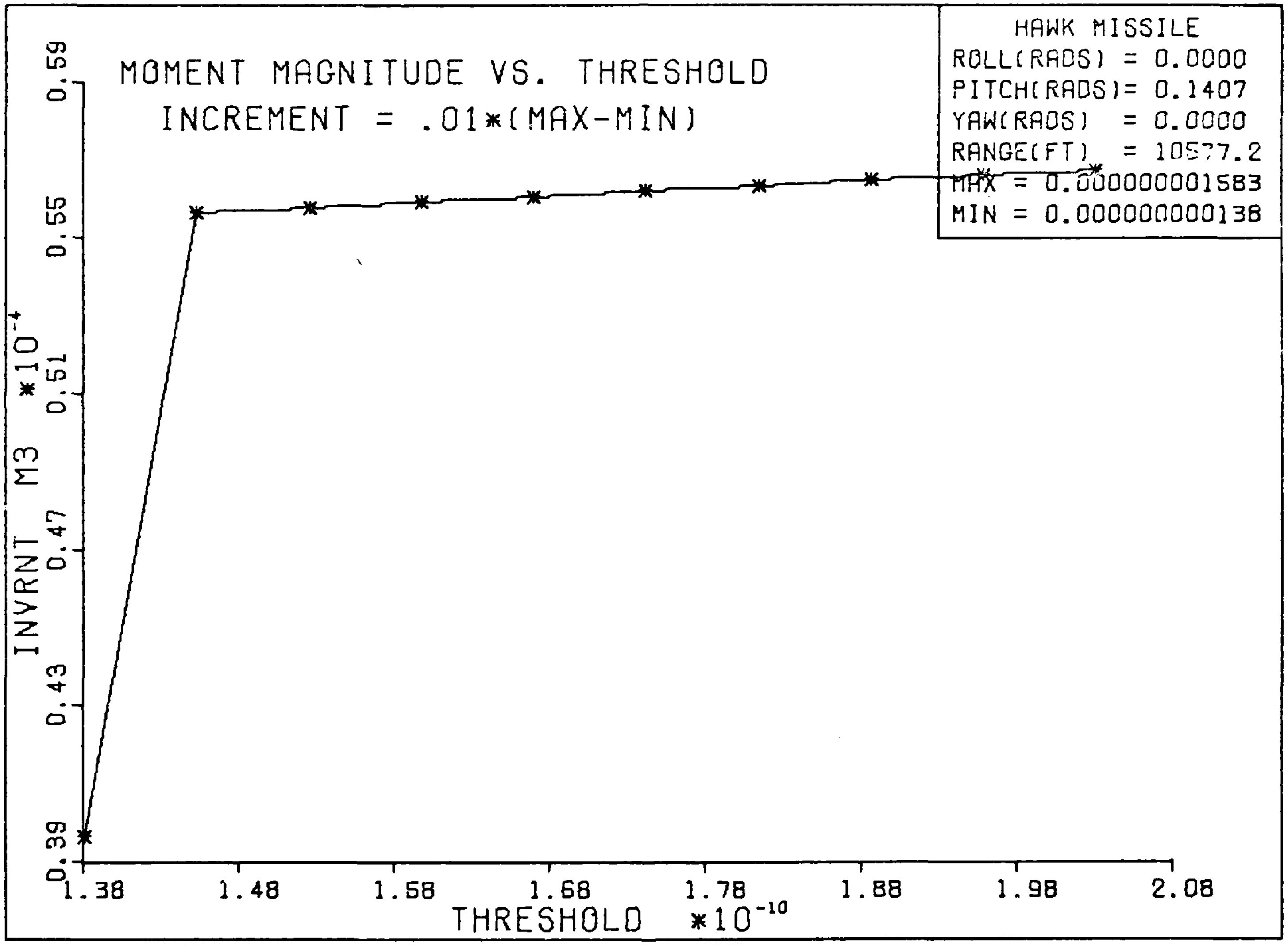

Figure 3. Variation of Moment Invariants versus Threshold.

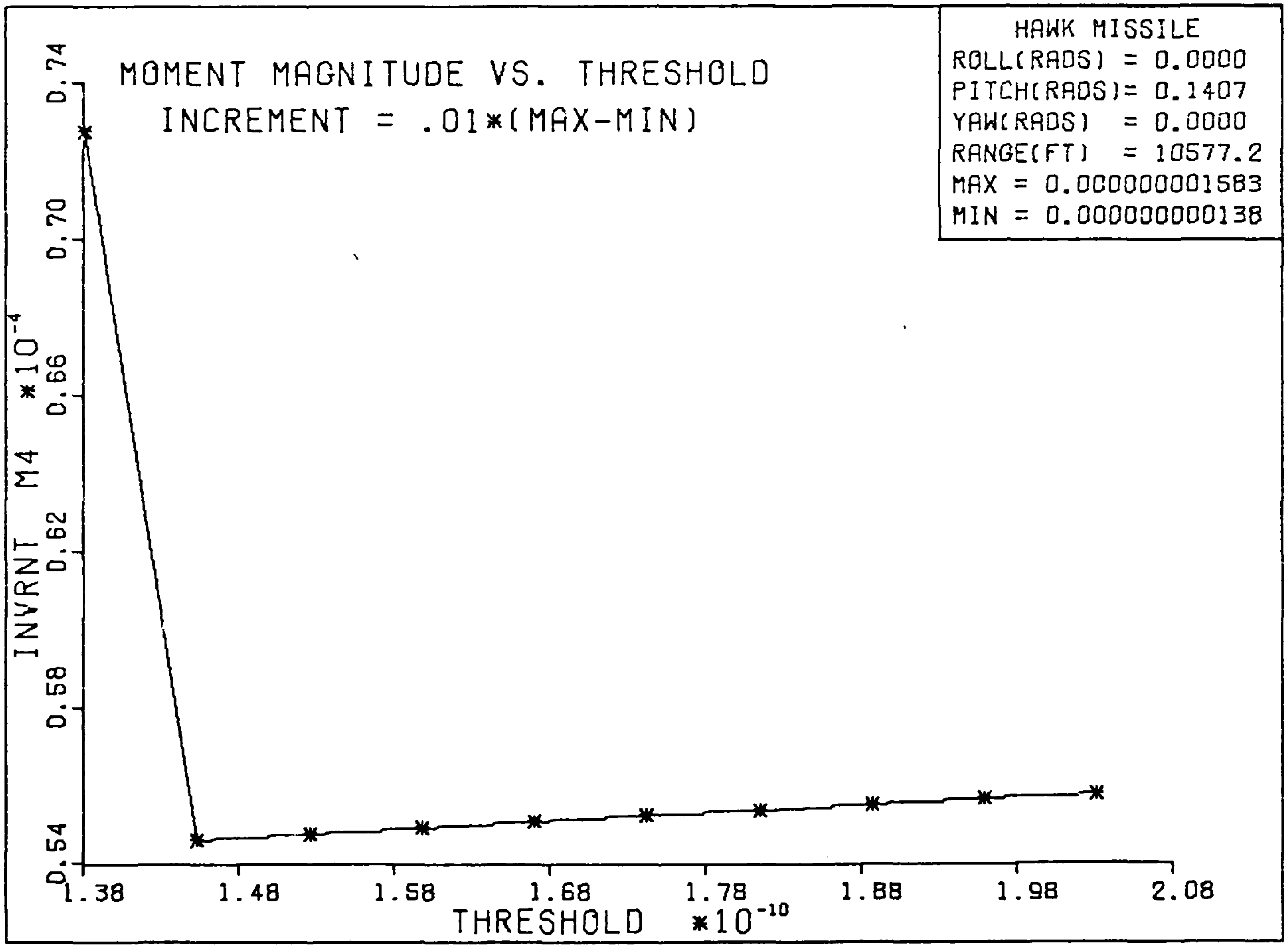

Figure 4. Variation of Moment Invariants versus Threshold.

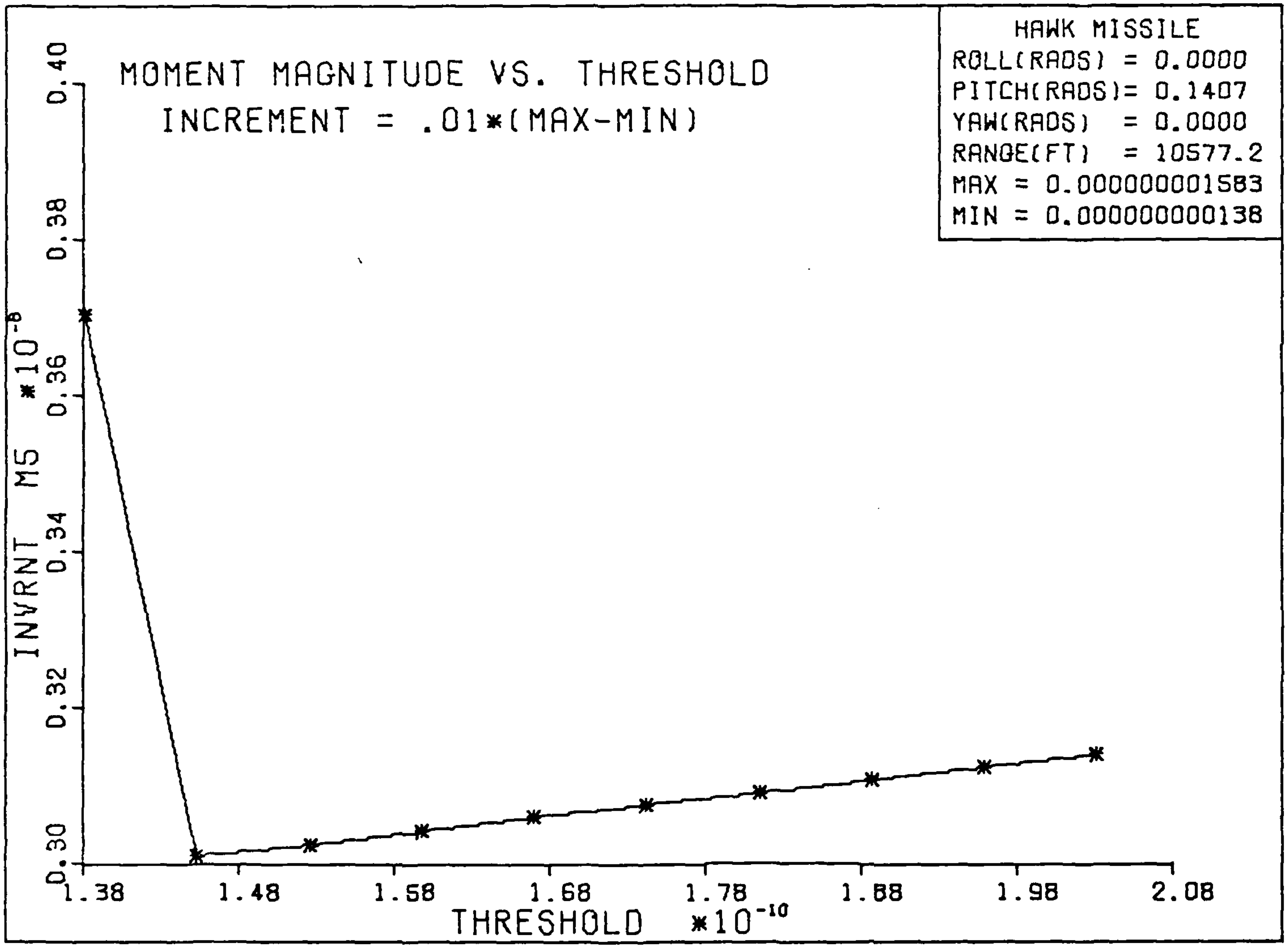

Figure 5. Variation of Moment Invariants versus Threshold.

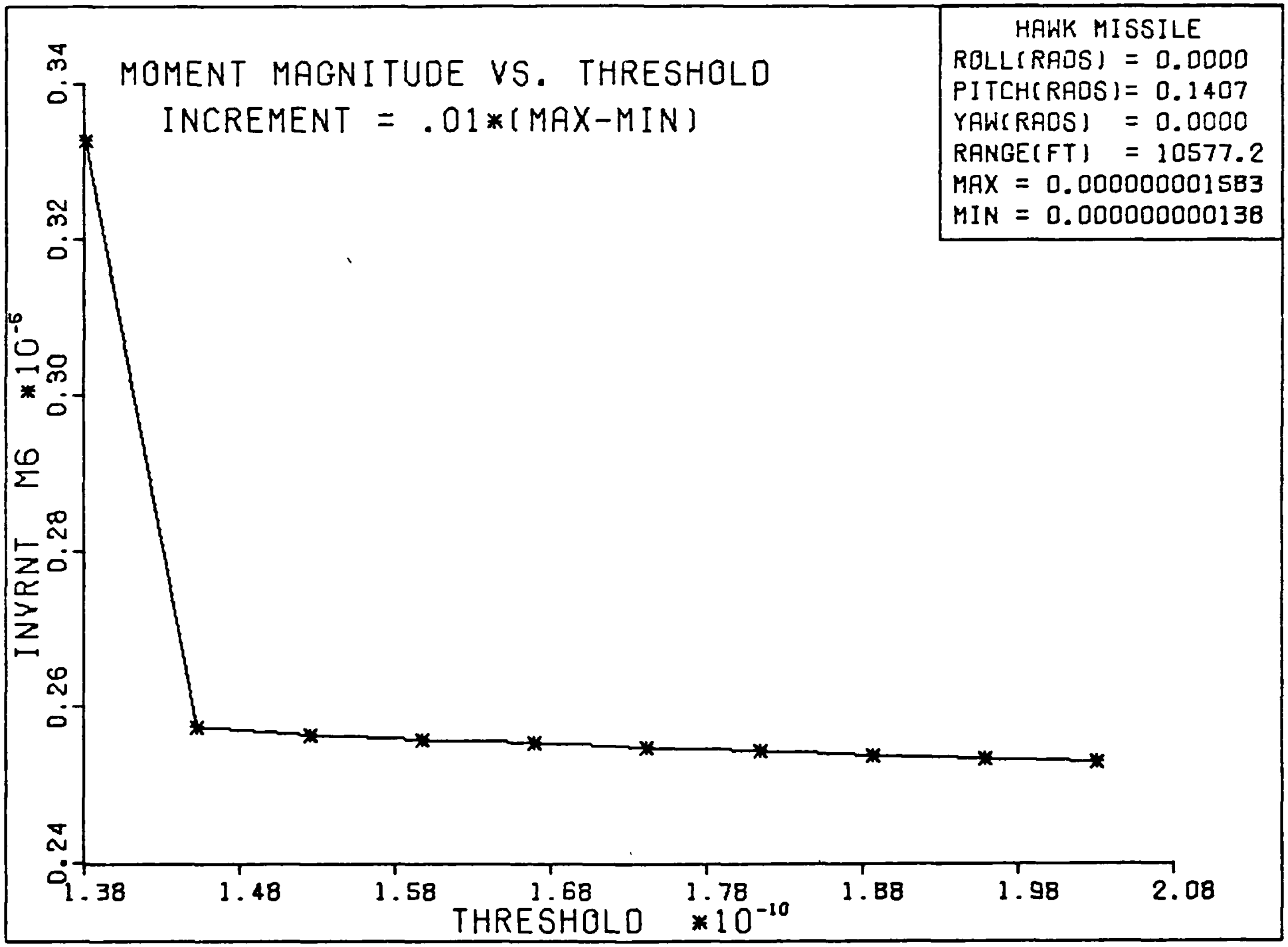

Figure 6. Variation of Moment Invariants versus Threshold.

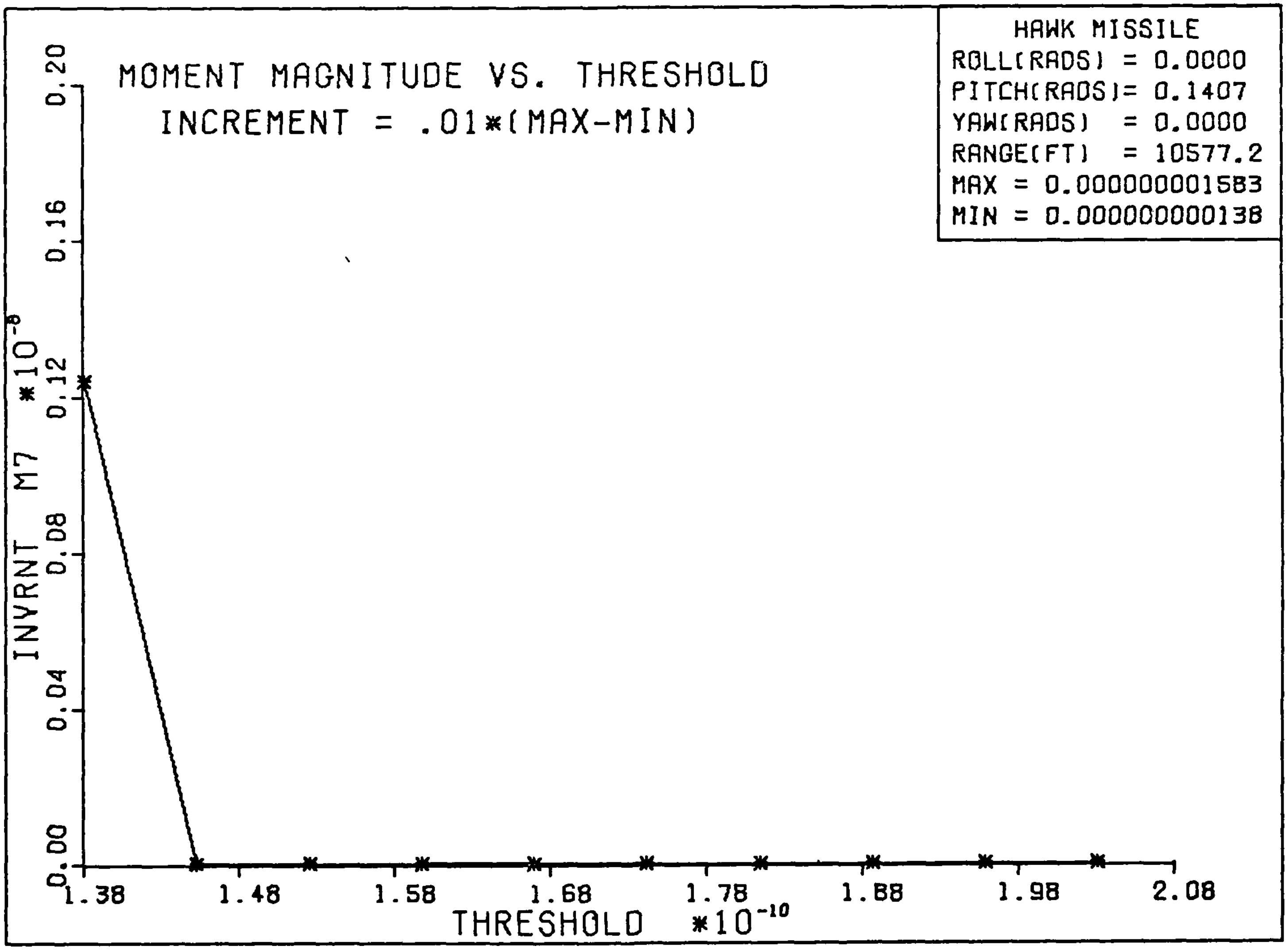

Figure 7. Variation of Moment Invariants versus Threshold.

The gradual variation over the latter points of the graphs indicates that the missile image is being affected.

Appendix C shows the variation of the raw and central moments versus threshold for the same case. The same observations can be made as above.

VI. Conclusions and Recommendations

In the preceding chapters, the general problem of how to identify an aircraft and its orientation was investigated. Specifically, the application of two-dimensional moment invariants was studied as a possible solution to the image pattern recognition problem. The concept of moment invariants, first introduced by Hu, was analyzed (pp.9-13) and provided a clever method for extracting a feature vector to describe an image by a number of quantities which are independent with respect to translation, similitude, and rotation.

Another set of moment invariants was derived (pp.27-36) by a completely different method based on the concepts of group theory applied to the two-dimensional moments of an image intensity distribution. A complete set of invariants under rotation was derived (pp.32-36) from which the image can be reconstructed. It was shown that Hu's invariants, obtained from the algebra of invariants, can be obtained from this set, and in fact, the two sets are totally equivalent. Thus, a full circle was completed, whereby the same set of invariants was derived through two complete techniques, algebraic invariants and group theory. However, the salient property of the group theoretical approach is that it provides a generalized technique to find invariants under other linear transformations. The

derivation by Hu cannot be extended readily to other transformations. A possible area of further investigation is to apply the group theory concepts to find invariants under the Seidel optical aberrations, coma and distortion.

The problem of background clutter in aircraft imagery data was addressed (pp.40-49). A threshold level was set depending on simple image parameters. Application to actual imagery data of a Hawk missile indicated that this may be an acceptable starting point for signal preprocessing.

Other image signal effects, including the aircraft engine plume and illumination variation, were discussed (pp.37-40). This area presents a wide field for further investigation into the statistical problems and methods to minimize the effects on pattern recognition.

This thesis provides a starting point for a follow-up study. Computer codes were written to efficiently calculate up to twentieth order raw moments and then to centralize them recursively (Appendix K). It is recommended that a laboratory mockup of various aircraft and an optical sensor system, video or others, be set up to provide imagery data for all aspect angles. From this data, a library of moment invariants can be constructed, and a statistical analysis can be performed. It is recommended that a classifier be implemented to identify target imagery data and the target orientation in order to determine to what order of moment invariants are required for reliable identification.

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Appendix A

Central Moments in Terms of Raw Moments

$$\mu_{pq} = \sum_{i=0}^{p+q} \sum_{j=S}^{T} (-1)^{p+q-j} \binom{p}{p-j}\binom{q}{q-i+j}$$

$$\cdot\ \bar{x}^{p-j}\ \bar{y}^{q-i+j}\ M_{j,i-j} \tag{A-1}$$

where $\bar{x} = M_{10}/M_{00}$ $\bar{y} = M_{01}/M_{00}$

$$\binom{a}{b} = \frac{a!}{b!\ (a-b)!} \quad \text{Binomial coefficients}$$

$$S = \frac{(i-q) + i-q}{2} \qquad T = \frac{(p+i) - p-i}{2}$$

$$\mu_{00} = M_{00} \tag{A-2}$$

$$\mu_{01} = -\bar{y}\ M_{00} + M_{01} = 0 \tag{A-3}$$

$$\mu_{01} = -\bar{x}\ M_{00} + M_{01} = 0$$

$$\mu_{02} = \bar{y}\ M_{00} - 2\bar{y}\ M_{01} + M_{02} \tag{A-4}$$

$$\mu_{11} = \bar{x}\ \bar{y}\ M_{00} - \bar{x}\ M_{01} - \bar{y}\ M_{10} + M_{11}$$

$$\mu_{20} = \bar{x}^{\ 2}\ M_{00} - 2\ \bar{x}\ M_{10} + M_{20}$$

$$\mu_{03} = -\bar{y}^3 M_{00} + 3\bar{y}^2 M_{01} - 3\bar{y}M_{02} + M_{03} \quad \text{(A-5)}$$

$$\mu_{12} = -\bar{x}\bar{y}^2 M_{00} + 2\bar{x}\bar{y} M_{01} + \bar{y}^2 M_{10} - \bar{x} M_{02}$$

$$-2\bar{y} M_{11} + M_{12}$$

$$\mu_{21} = -\bar{x}^2\bar{y} M_{00} + \bar{x}^2 M_{01} + 2\bar{x}\bar{y} M_{10} - 2\bar{x} M_{11}$$

$$-\bar{y} M_{20} + M_{21}$$

$$\mu_{30} = -\bar{x}^3 M_{00} + 3\bar{x}^2 M_{10} - 3\bar{x} M_{20} + M_{30}$$

$$\mu_{04} = \bar{y}^4 M_{00} - 4\bar{y}^3 M_{01} + 6\bar{y}^2 M_{02} - 4\bar{y} M_{03} + M_{04} \quad \text{(A-6)}$$

$$\mu_{13} = \bar{x}\bar{y}^3 M_{00} - 3\bar{x}\bar{y}^2 M_{01} - \bar{y}^3 M_{10} + 3\bar{y}^2 M_{11} \bar{x} M_{03}$$

$$-3\bar{y} M_{12} + M_{13}$$

$$\mu_{22} = \bar{x}^2\bar{y}^2 M_{00} - 2\bar{x}^2\bar{y} M_{01} - 2\bar{x}\bar{y}^2 M_{10} + \bar{x}^2 M_{02}$$

$$+4\bar{x}\bar{y} M_{11} + \bar{y}^2 M_{20} - 2\bar{x} M_{12} - 2\bar{y} M_{21} + M_{22}$$

$$\mu_{31} = \bar{x}^3\bar{y} M_{00} - \bar{x}^3 M_{01} - 3\bar{x}^2\bar{y} M_{10} + 3\bar{x}^2 M_{11} + 3\bar{x}\bar{y} M_{20}$$

$$-3\bar{x} M_{21} - \bar{y} M_{30} + M_{31}$$

$$\mu_{40} = \bar{x}^4 M_{00} - 4\bar{x}^3 M_{10} + 6\bar{x}^2 M_{20} - 4\bar{x} M_{30} + M_{40}$$

$$\mu_{05} = -\bar{y}^5 M_{00} + 5\bar{y}^4 M_{01} - 10\bar{y}^3 M_{02} + 10\bar{y}^2 M_{03}$$

$$-5\bar{y} M_{04} + M_{05} \tag{A-7}$$

$$\mu_{14} = -\bar{x}\bar{y}^4 M_{00} + 4\bar{x}\bar{y}^3 M_{01} + \bar{y}^4 M_{10} - 6\bar{x}\bar{y}^2 M_{02}$$

$$-4\bar{y}^3 M_{11} + 4\bar{x}\bar{y} M_{03} + 6\bar{y}^2 M_{12} - \bar{x} M_{04}$$

$$-4\bar{y} M_{13} + M_{14}$$

$$\mu_{23} = -\bar{x}^2\bar{y}^3 M_{00} - 3\bar{x}^2\bar{y}^2 M_{01} + 2\bar{x}\bar{y}^3 M_{10} - 3\bar{x}^2\bar{y} M_{02}$$

$$-6\bar{x}\bar{y}^2 M_{11} - \bar{y}^3 M_{20} + \bar{x}^2 M_{03} + 6\bar{x}\bar{y} M_{12}$$

$$+3\bar{y}^2 M_{21} - 2\bar{x} M_{13} - 3\bar{y} M_{22} + M_{23}$$

$$\mu_{32} = -\bar{x}^3\bar{y}^2 M_{00} + 2\bar{x}^3\bar{y} M_{01} + 3\bar{x}^2\bar{y}^2 M_{10} - \bar{x}^3 M_{02}$$

$$-6\bar{x}^2\bar{y} M_{11} - 3\bar{x}\bar{y}^2 M_{20} + 3\bar{x}^2 M_{12} + 6\bar{x}\bar{y} M_{21}$$

$$+\bar{y}^2 M_{30} - 3\bar{x} M_{22} - 2\bar{y} M_{31} + M_{32}$$

$$\mu_{41} = -\bar{x}^4 \bar{y} M_{00} + \bar{x}^4 M_{01} + 4 \bar{x}^3 \bar{y} M_{10} - 4 \bar{x}^3 M_{11}$$

$$-6 \bar{x}^2 \bar{y} M_{20} + 6 \bar{x}^2 M_{21} + 4 \bar{x} \bar{y} M_{20} - 4 \bar{x} M_{31}$$

$$-\bar{y} M_{40} + M_{41}$$

$$\mu_{50} = -\bar{x}^5 M_{00} + 5 \bar{x}^4 M_{10} - 10 \bar{x}^3 M_{20} + 10 x^2 M_{30}$$

$$-5 \bar{x} M_{40} x M_{50}$$

Appendix B

Recursive Central Moments

$$\mu_{pq} = M_{pq} \sum_{i=0}^{p+q-1} \sum_{j=S}^{I} \binom{p}{p-j} \binom{q}{q-i+j}$$

$$\cdot\ \bar{x}^{p-j}\ \bar{y}^{q-i+j} \cdot \mu_{j,\ i-j} \qquad \text{(B-1)}$$

where $\bar{x} = M_{10}/M_{00}$ $\quad y = M_{01}/M_{00}$

$$S = \frac{(i-q) + |i-q|}{2} \qquad T = \frac{(p+i) - |p-i|}{2}$$

$$\binom{a}{b} = \frac{a!}{b!\,(a-b)!} \quad \text{Binomial coefficients}$$

$$\mu_{00} = M_{00} \qquad \text{(B-2)}$$

$$\mu_{01} = -\bar{y}\ \mu_{00} + M_{01} = 0$$

$$\mu_{10} = \bar{x}\ \mu_{00} + M_{10} = 0 \qquad \text{(B-3)}$$

$$\mu_{02} = -\bar{y}^2 \mu_{00} - 2\bar{y}\ \mu_{01} + M_{02}$$

$$\mu_{11} = -\bar{x}\ \bar{y}\ \mu_{00} - \bar{x}\ \mu_{01} - \bar{y}\ \mu_{10} + M_{11}$$

$$\mu_{20} = -\bar{x}^2\ \mu_{00} - 2\ \bar{x}\ \mu_{10} + M_{20} \qquad \text{(B-4)}$$

$$\mu_{03} = -\bar{y}^3 \mu_{00} - 3\bar{y}^2 \mu_{01} - 3\bar{y}\mu_{02} + M_{03}$$

$$\mu_{12} = -\bar{x}\bar{y}^2\mu_{00} - 2\bar{x}\bar{y}\mu_{01} - \bar{y}^2\mu_{10} - \bar{x}\mu_{02} - 2\bar{y}\mu_{11} + M_{12}$$

$$\mu_{21} = -\bar{x}^2\bar{y}\mu_{00} - \bar{x}^2\mu_{01} - 2\bar{x}\bar{y}\mu_{10} - 2\bar{x}\mu_{11} - \bar{y}\mu_{20} + M_{21}$$

$$\mu_{30} = -\bar{x}^3\mu_{00} - 3\bar{x}^2\mu_{10} - 3\bar{x}\mu_{20} + M_{30} \qquad \text{(B-5)}$$

$$\mu_{04} = -\bar{y}^4\mu_{00} - 4\bar{y}^3\mu_{01} - 6\bar{y}^2\mu_{02} - 4\bar{y}\mu_{03} + M_{04}$$

$$\mu_{13} = -\bar{x}\bar{y}^3\mu_{00} - 3\bar{x}\bar{y}^2\mu_{01} - \bar{y}^3\mu_{10} - 3\bar{x}\bar{y}\mu_{02} - 3\bar{y}^2\mu_{11}$$

$$- \bar{x}\mu_{03} - 3\bar{y}\mu_{12} + M_{13}$$

$$\mu_{22} = -\bar{x}^2\bar{y}^2\mu_{00} - 2\bar{x}^2\bar{y}\mu_{01} - 2\bar{x}\bar{y}^3\mu_{10} - \bar{x}^2\mu_{00} - 4\bar{x}\bar{y}\mu_{11}$$

$$- \bar{y}^2\mu_{20} - 2\bar{x}\mu_{12} - 2\bar{y}\mu_{21} + M_{22}$$

$$\mu_{31} = -\bar{x}^3\bar{y}\mu_{00} - \bar{x}^3\mu_{01} - 3\bar{x}^2\bar{y}\mu_{10} - 3\bar{x}^2\mu_{11} - 3\bar{x}\bar{y}\mu_{20}$$

$$- 3\bar{x}\mu_{21} - \bar{y}\mu_{30} + M_{31}$$

$$\mu_{40} = \bar{x}^4\mu_{00} - 4\bar{x}^3\mu_{10} - 6\bar{x}^2\mu_{20} - 4\bar{x}\mu_{30} + M_{40} \qquad \text{(B-6)}$$

$$\mu_{05} = -\bar{y}^5\mu_{00} - 5\bar{y}^4\mu_{01} - 10\bar{y}^3\mu_{02} - 10\bar{y}^2\mu_{03} - 5\bar{y}\mu_{04} + M_{05}$$

$$\mu_{14} = -\bar{x}\,\bar{y}^4\,\mu_{00} -4\,\bar{x}\,\bar{y}^3\,\mu_{01} -\bar{y}^4\mu_{10} -6\,\bar{x}\,\bar{y}^2\,\mu_{02} -4\,\bar{y}^3\,\mu_{11}$$

$$-4\,\bar{x}\,\bar{y}\,\mu_{03} -6\,\bar{y}^2\,\mu_{12} -\bar{x}\,\mu_{04} -4\,\bar{y}\,\mu_{13} +M_{14}$$

$$\mu_{23} = -\bar{x}^2\bar{y}^3\,\mu_{00} -3\,\bar{x}^2\bar{y}^2\,\mu_{01} -2\,\bar{x}\,\bar{y}^3\,\mu_{10} -3\,\bar{x}^2\bar{y}\,\mu_{02}$$

$$-6\,\bar{x}\,\bar{y}\,\mu_{11} -\bar{y}^3\mu_{20} -\bar{x}^2\,\mu_{03} -6\,\bar{x}\,\bar{y}\,\mu_{12} -3\,\bar{y}^2\mu_{21}$$

$$-2\,\bar{x}\,\mu_{13} -3\,\bar{y}\,\mu_{22} +M_{23}$$

$$\mu_{32} = -\bar{x}^3\bar{y}^2\mu_{00} -2\,\bar{x}^3\bar{y}\,\mu_{01} -3\,\bar{x}^2\bar{y}^2\,\mu_{10} -\bar{x}^3\mu_{02} -6\,\bar{x}^2\bar{y}\,\mu_{11}$$

$$-3\,\bar{x}\,\bar{y}^2\mu_{20} -3\,\bar{x}^2\mu_{12} -6\,\bar{x}\,\bar{y}\,\mu_{21} -\bar{y}^2\mu_{30} -3\,\bar{x}\,\mu_{22}$$

$$-2\,\bar{y}\,\mu_{31} +M_{32}$$

$$\mu_{41} = -\bar{x}^4\bar{y}\,\mu_{00} -\bar{x}^4\mu_{01} -4\,\bar{x}^3\bar{y}\,\mu_{10} -4\,\bar{x}^3\mu_{11} -6\,\bar{x}^2\bar{y}\,\mu_{20}$$

$$-6\,\bar{x}^2\mu_{21} -4\,\bar{x}\,\bar{y}\,\mu_{30} -4\,\bar{x}\,\mu_{31} -\bar{y}\,\mu_{40} +M_{41}$$

$$\mu_{50} = -\bar{x}^5\mu_{00} -5\,\bar{x}^4\mu_{10} -10\,\bar{x}^3\,\mu_{20} -10\,\bar{x}^2\,\mu_{30} -5\bar{x}\,\mu_{40} +M_{50} \tag{B-7}$$

Appendix C

Additional Data of Threshold Analysis

This appendix contains additional data pertaining to the target background thresholding problem in Chapter IV. Table C-1 lists numerical values of the raw moments through the third order as the threshold was varied. Figures C-1 through C-10 show the same data in graphical form. Table C-2 and Figures C-11 through C-20 illustrate the same process, but for the central moments.

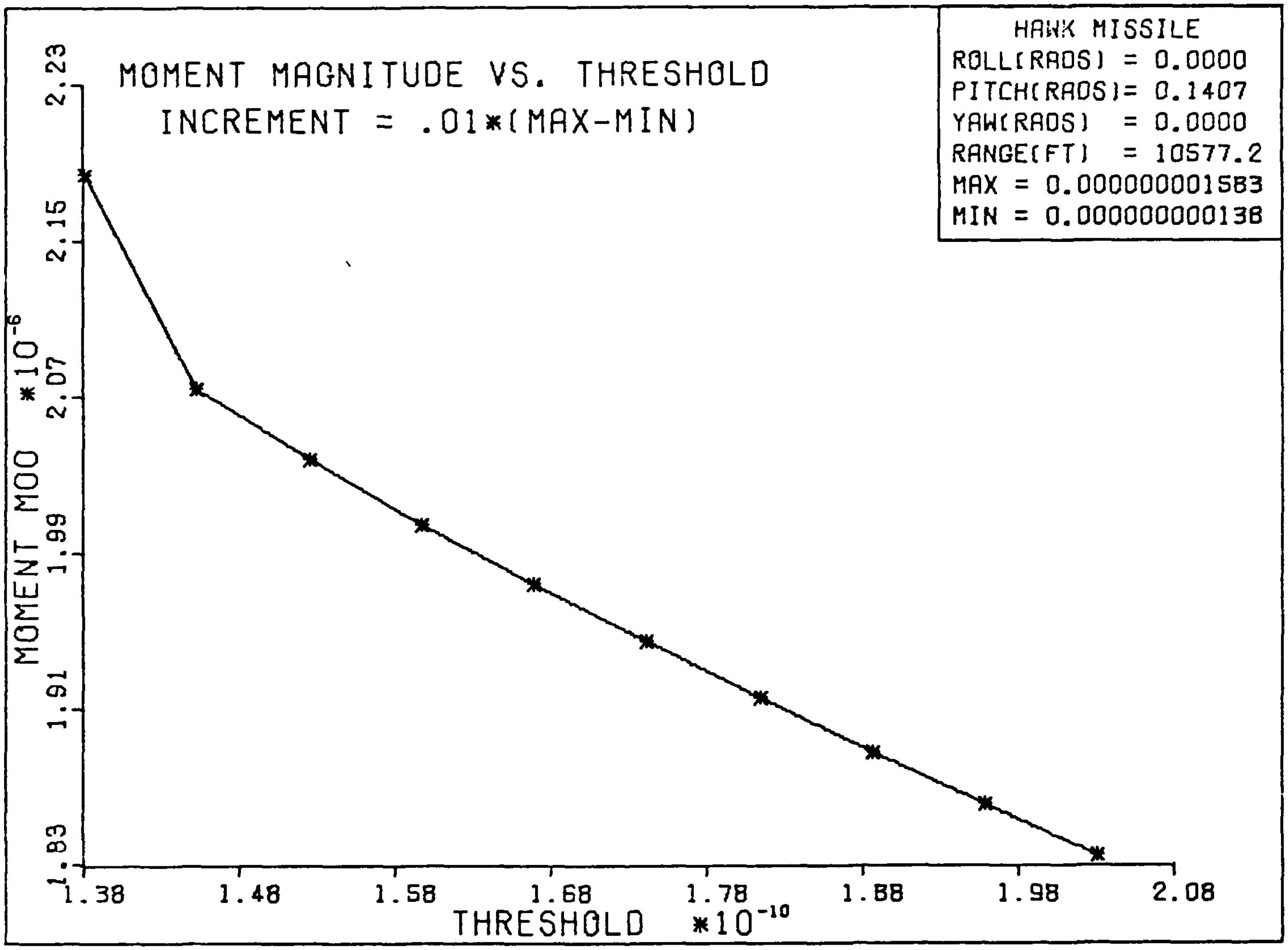

Figure C-1. Variation of Raw Moment M_{00} versus Threshold.

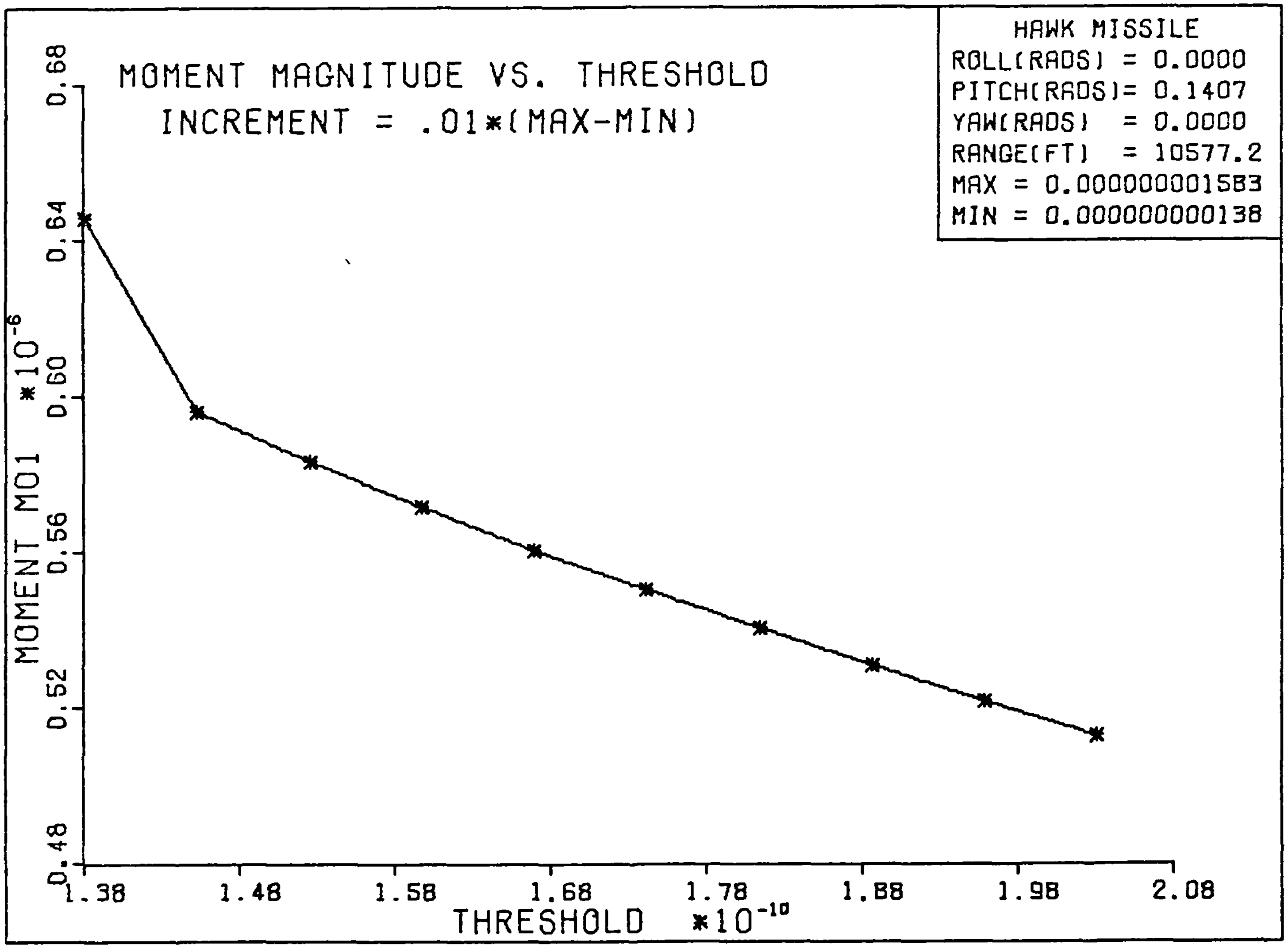

Figure C-2. Variation of Raw Moment M_{01} versus Threshold.

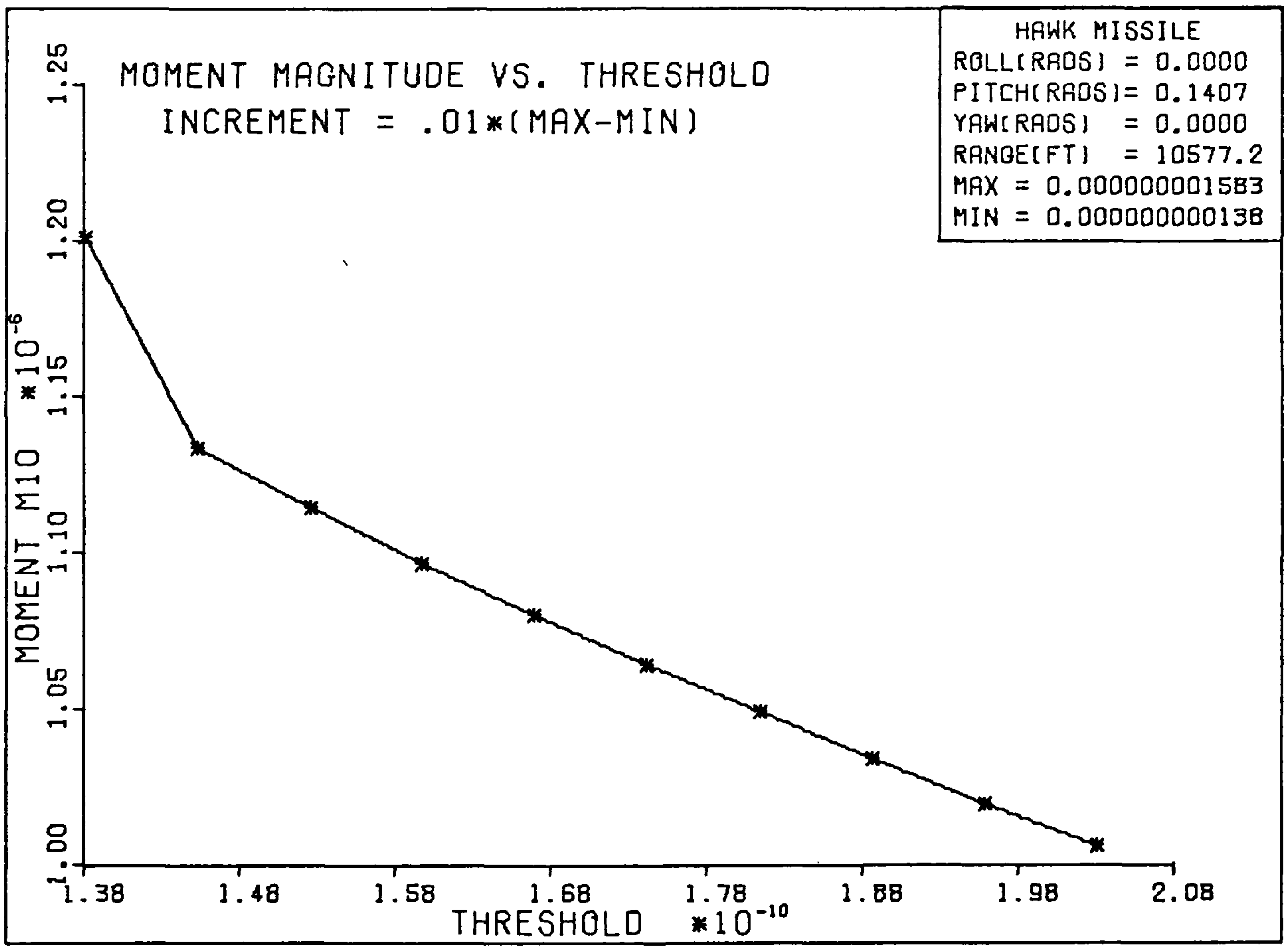

Figure C-3. Variation of Raw Moment M_{10} versus Threshold.

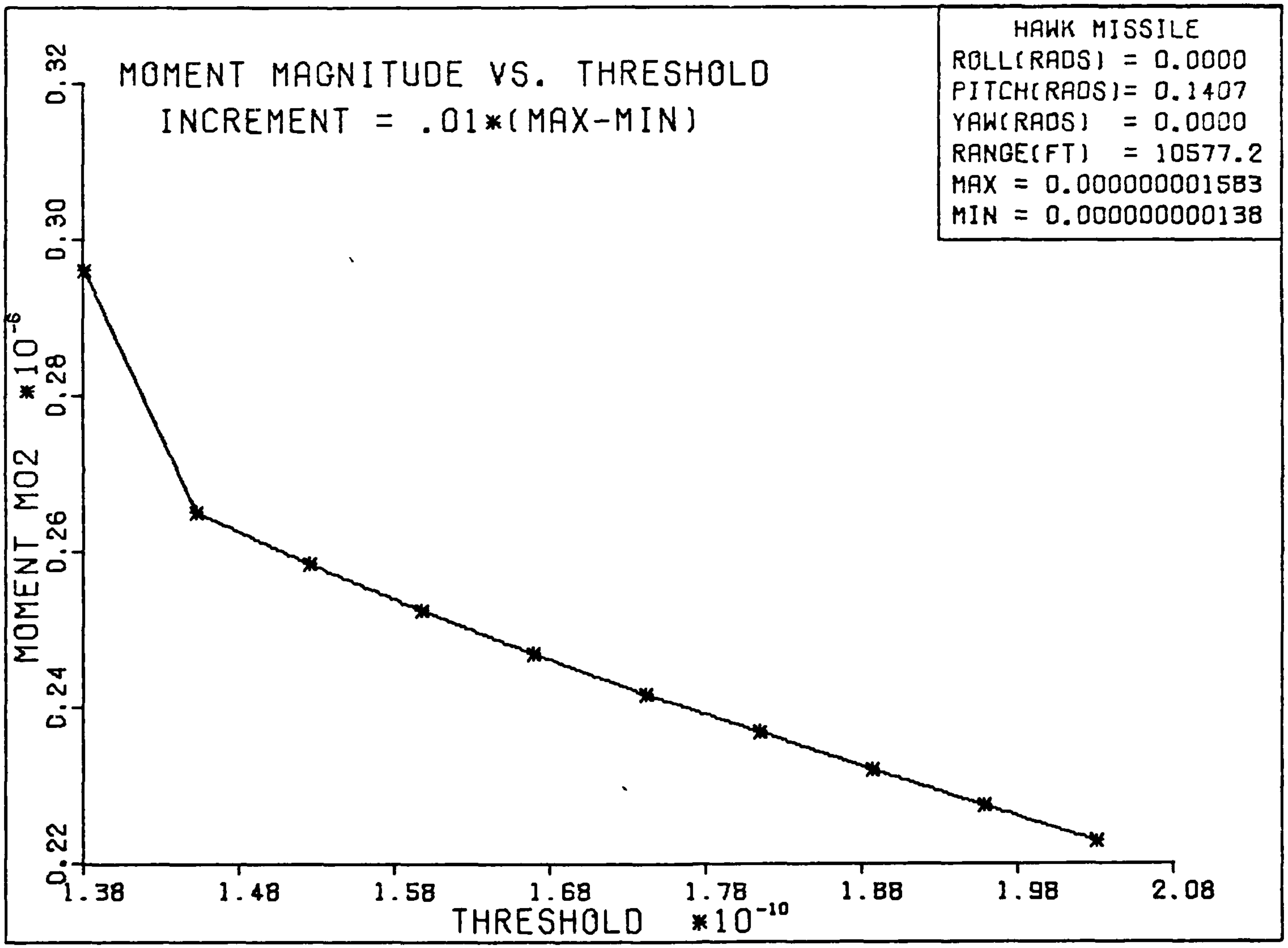

Figure C-4. Variation of Raw Moment M_{02} versus Threshold.

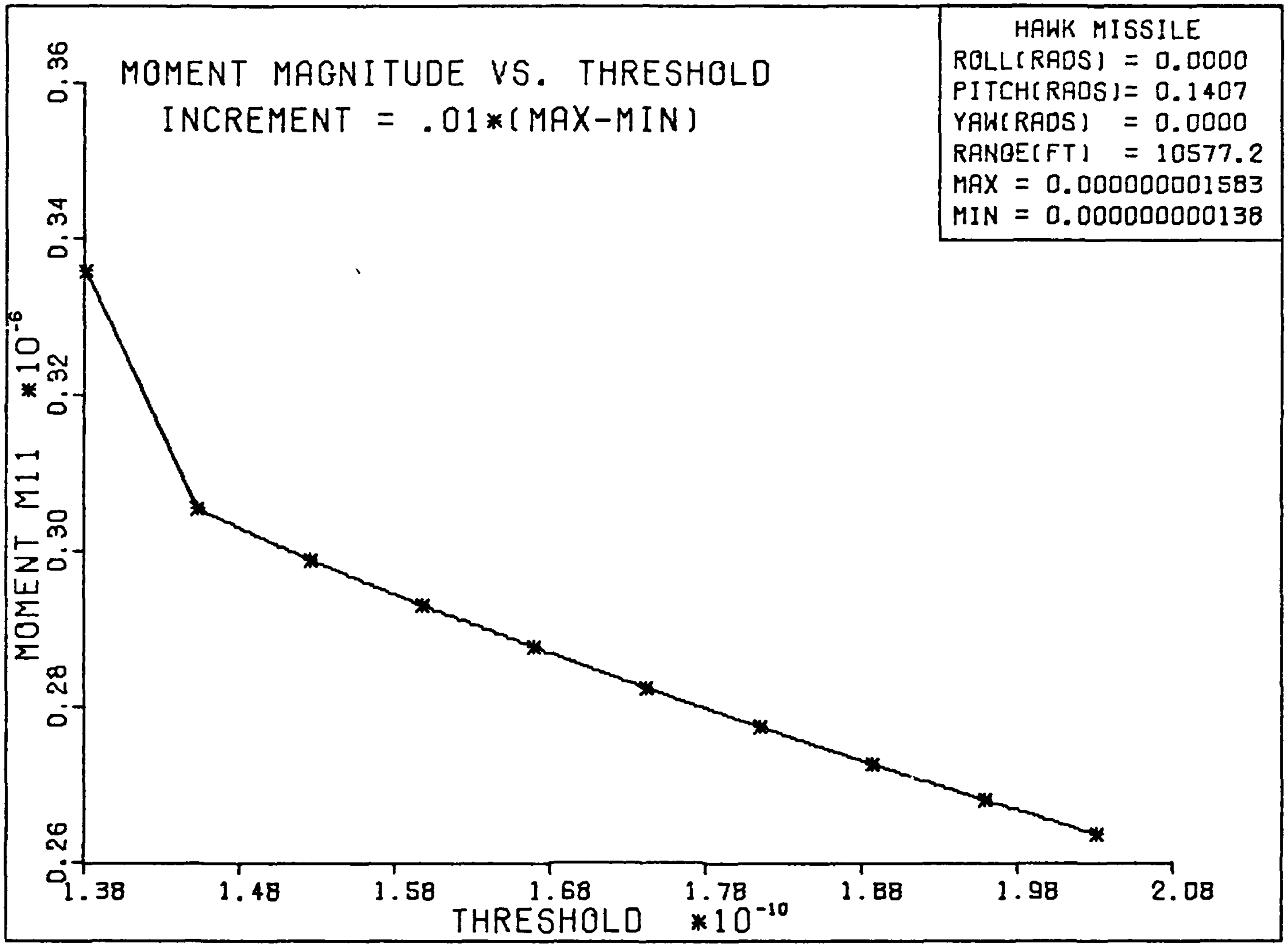

Figure C-5. Variation of Raw Moment M_{11} versus Threshold.

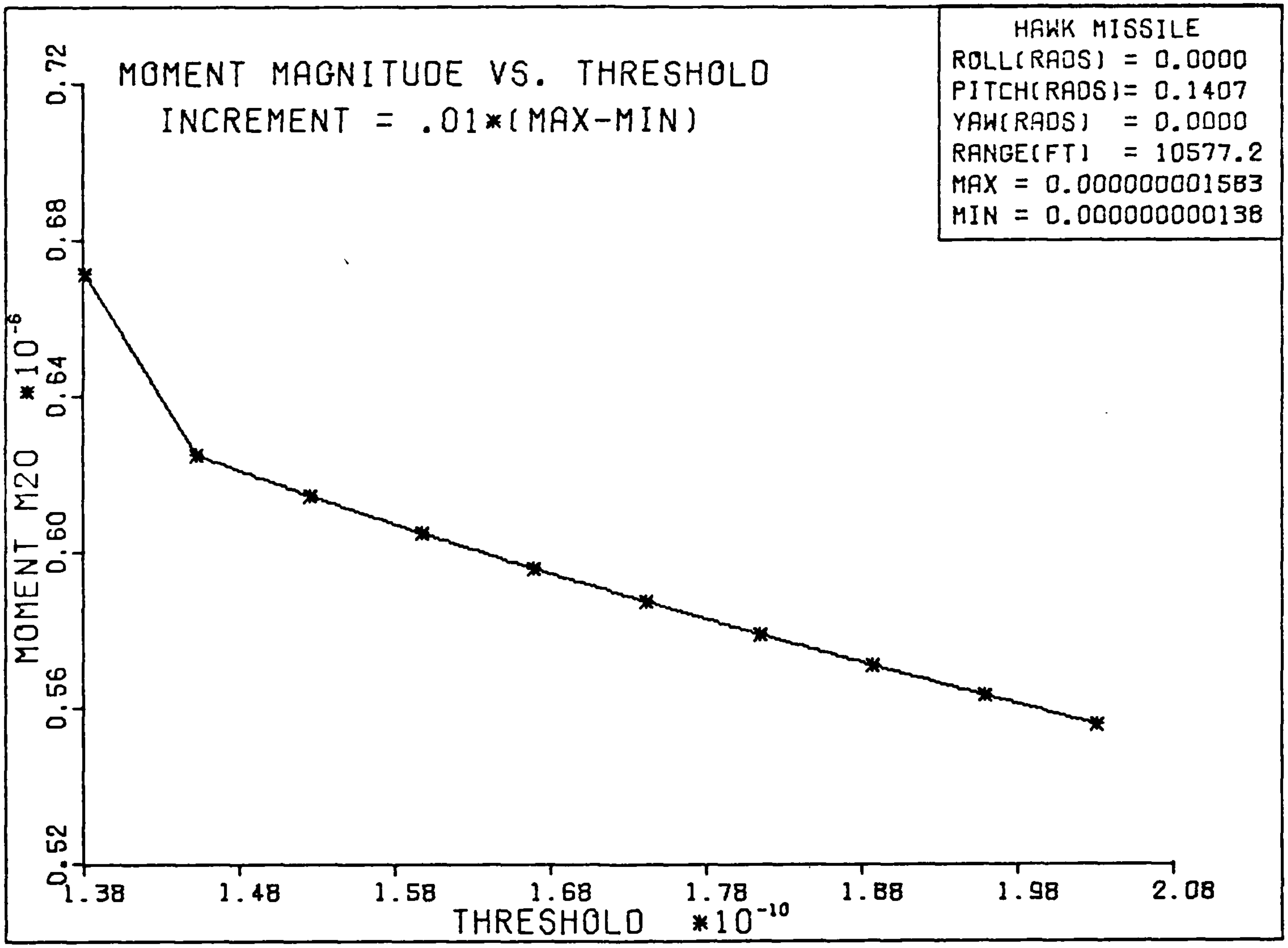

Figure C-6. Variation of Raw Moment M_{20} versus Threshold.

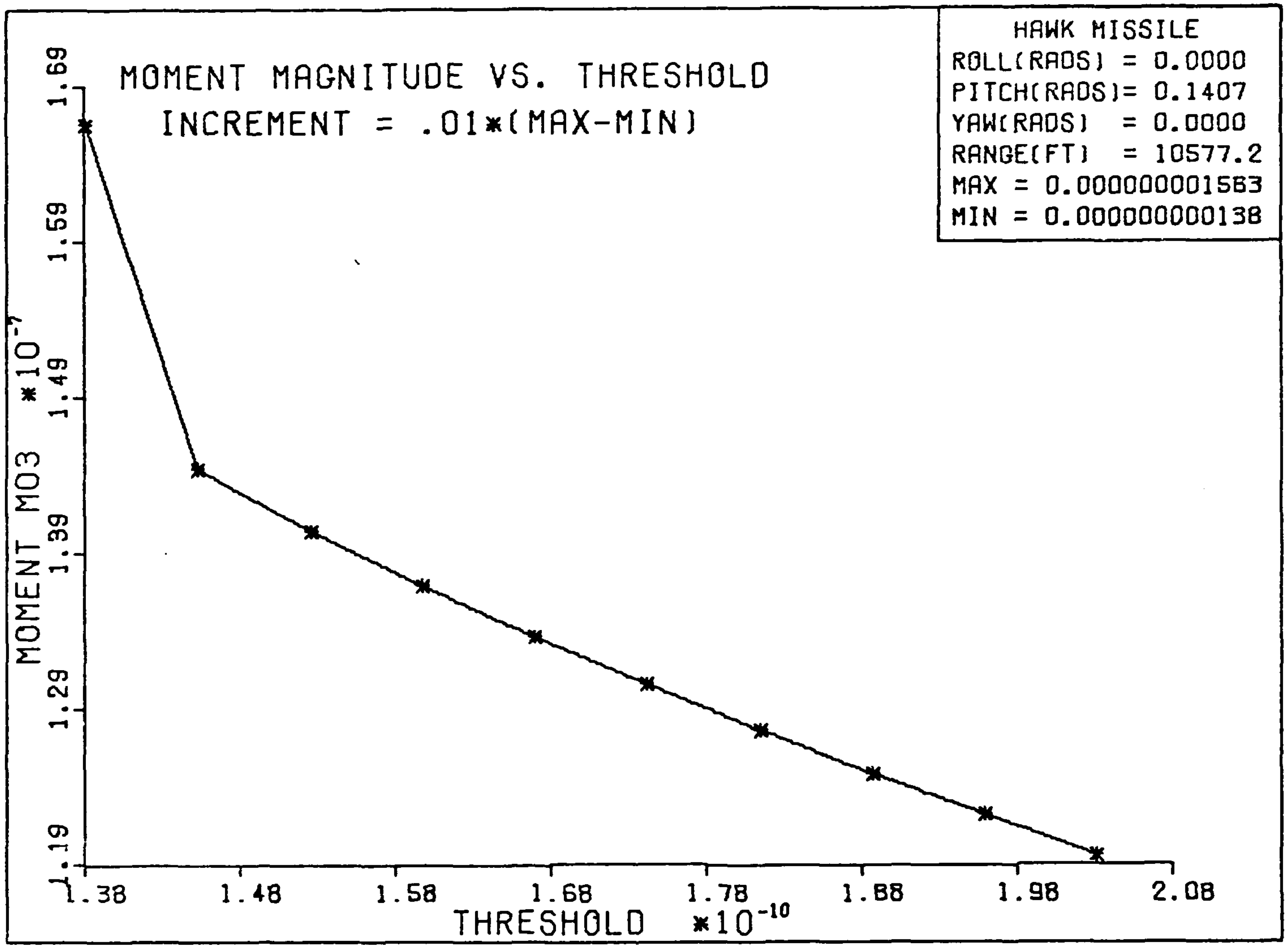

Figure C-7. Variation of Raw Moment M_{03} versus Threshold.

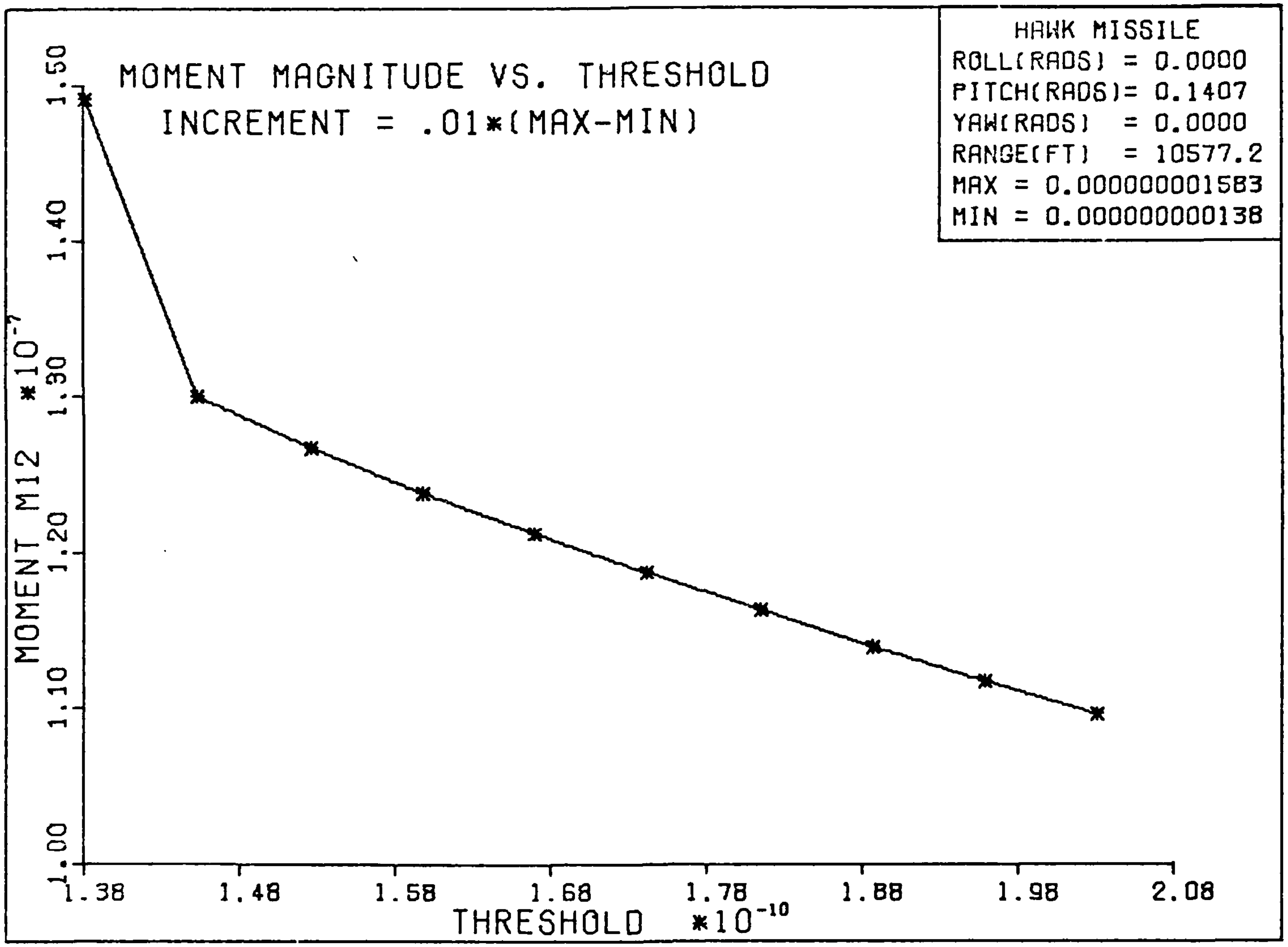

Figure C-8. Variation of Raw Moment M_{12} versus Threshold.

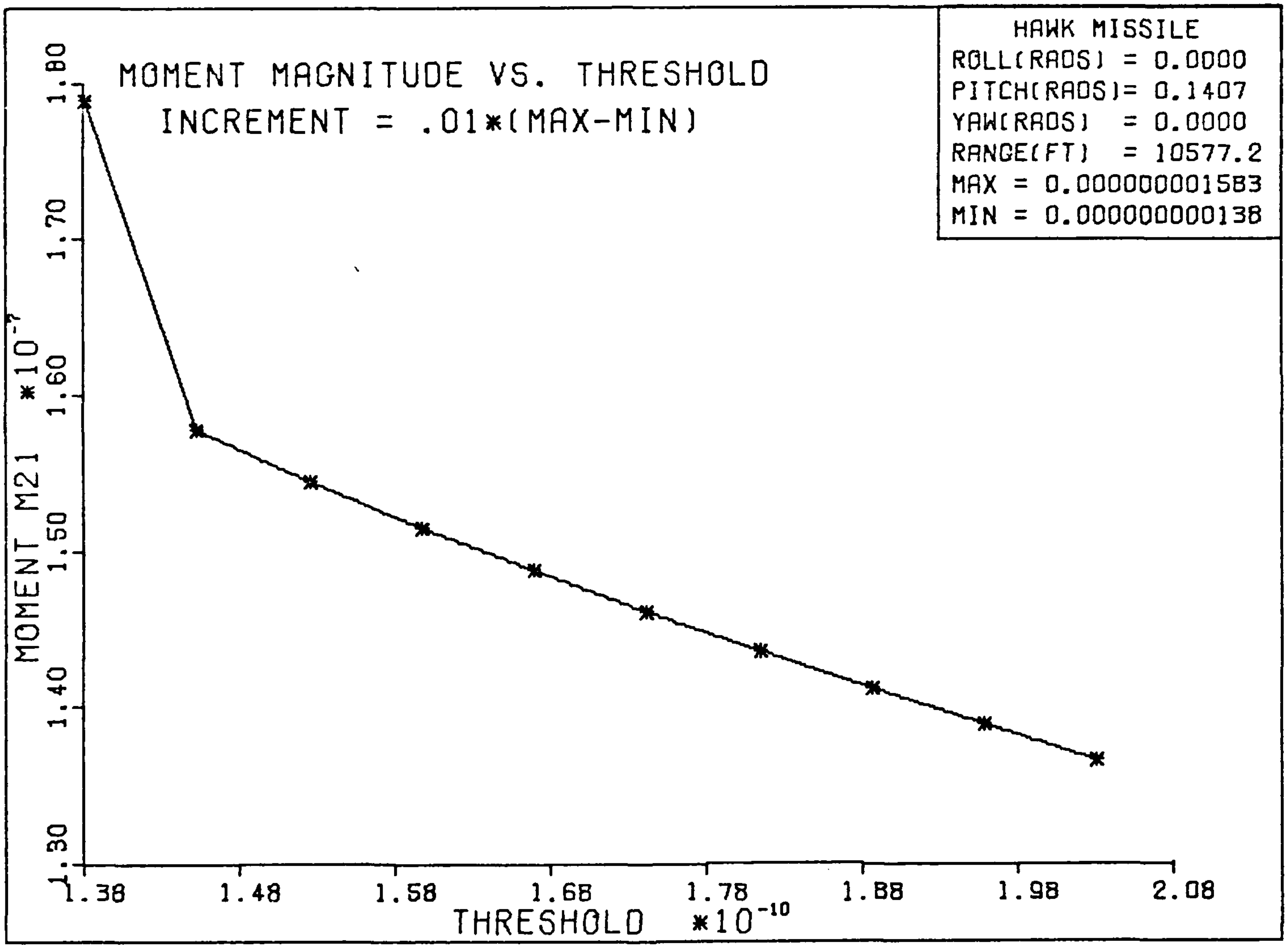

Figure C-9. Variation of Raw Moment M_{21} versus Threshold.

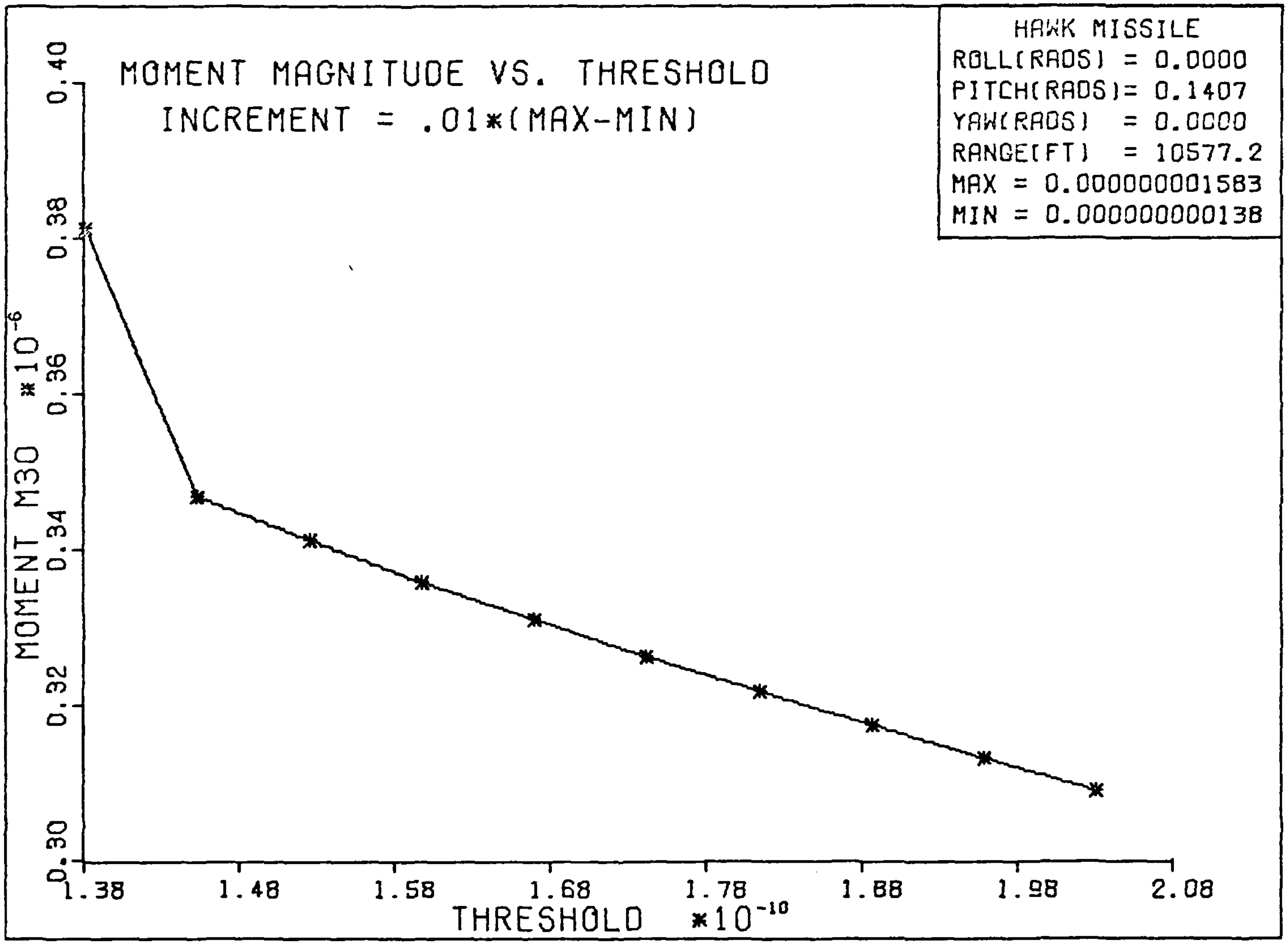

Figure C-10. Variation of Raw Moment M_{30} versus Threshold.

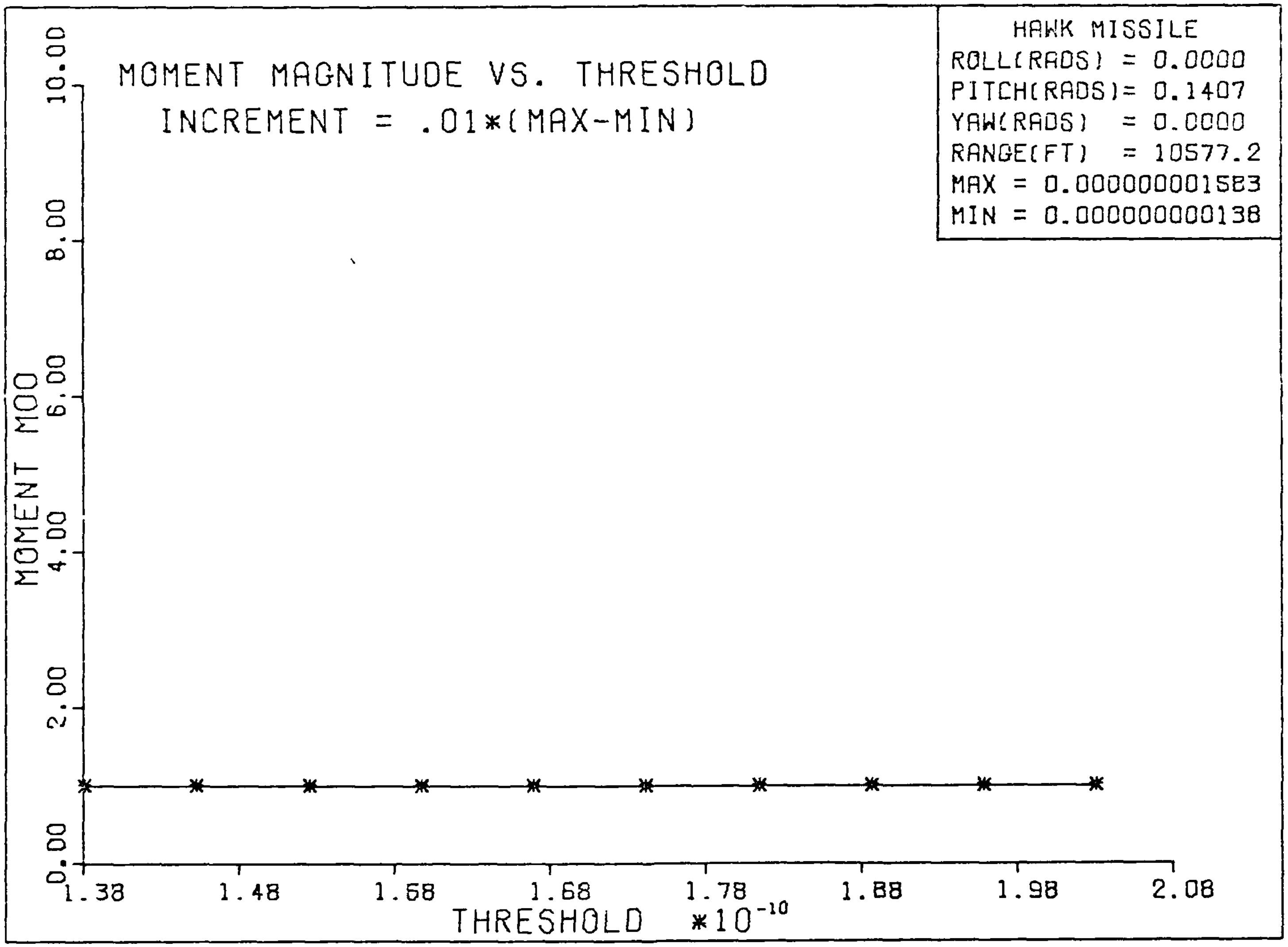

Figure C-11. Variation of Central Moment μ_{00} versus Threshold.

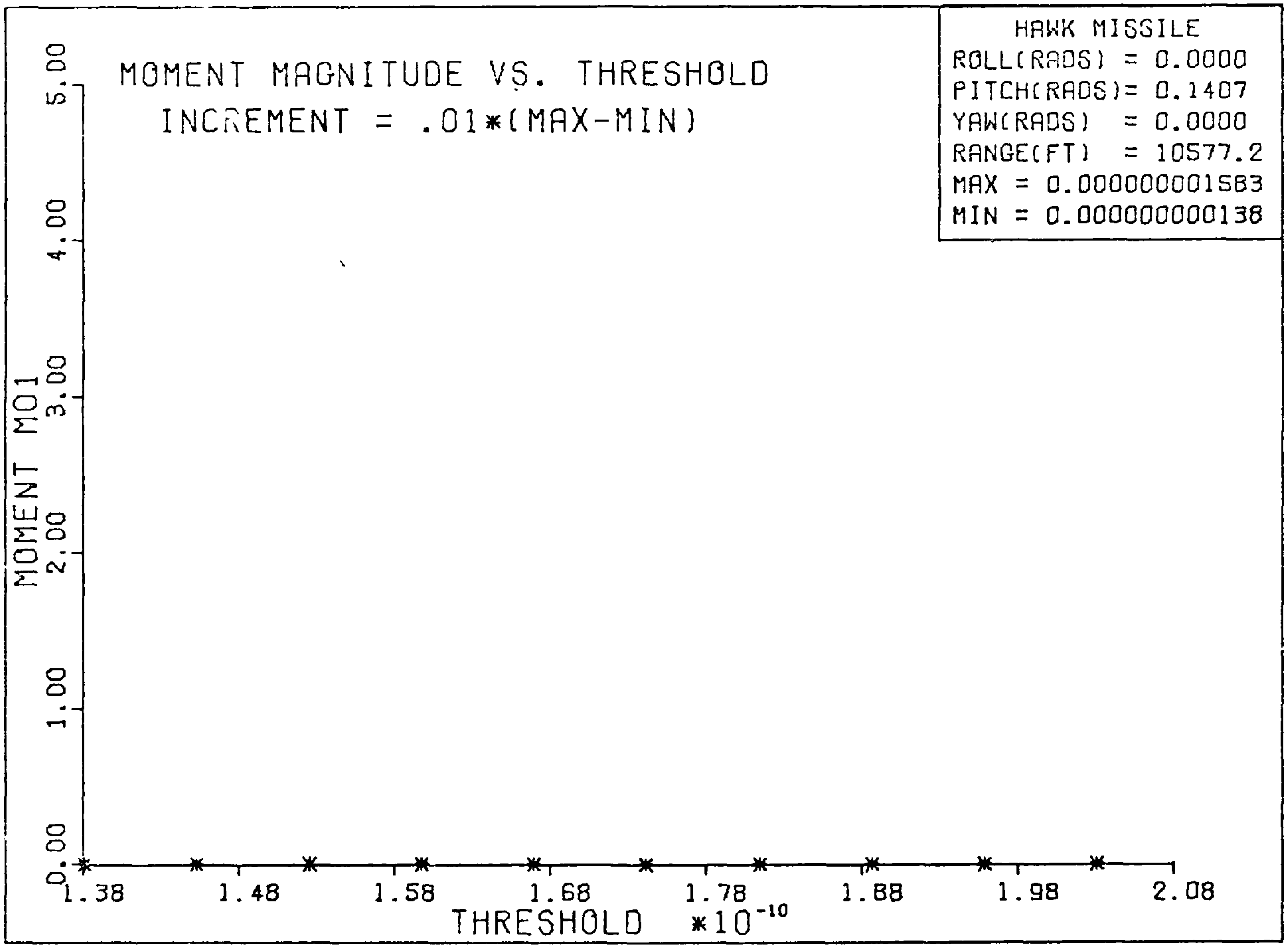

Figure C-12. Variation of Central Moment μ_{01} versus Threshold.

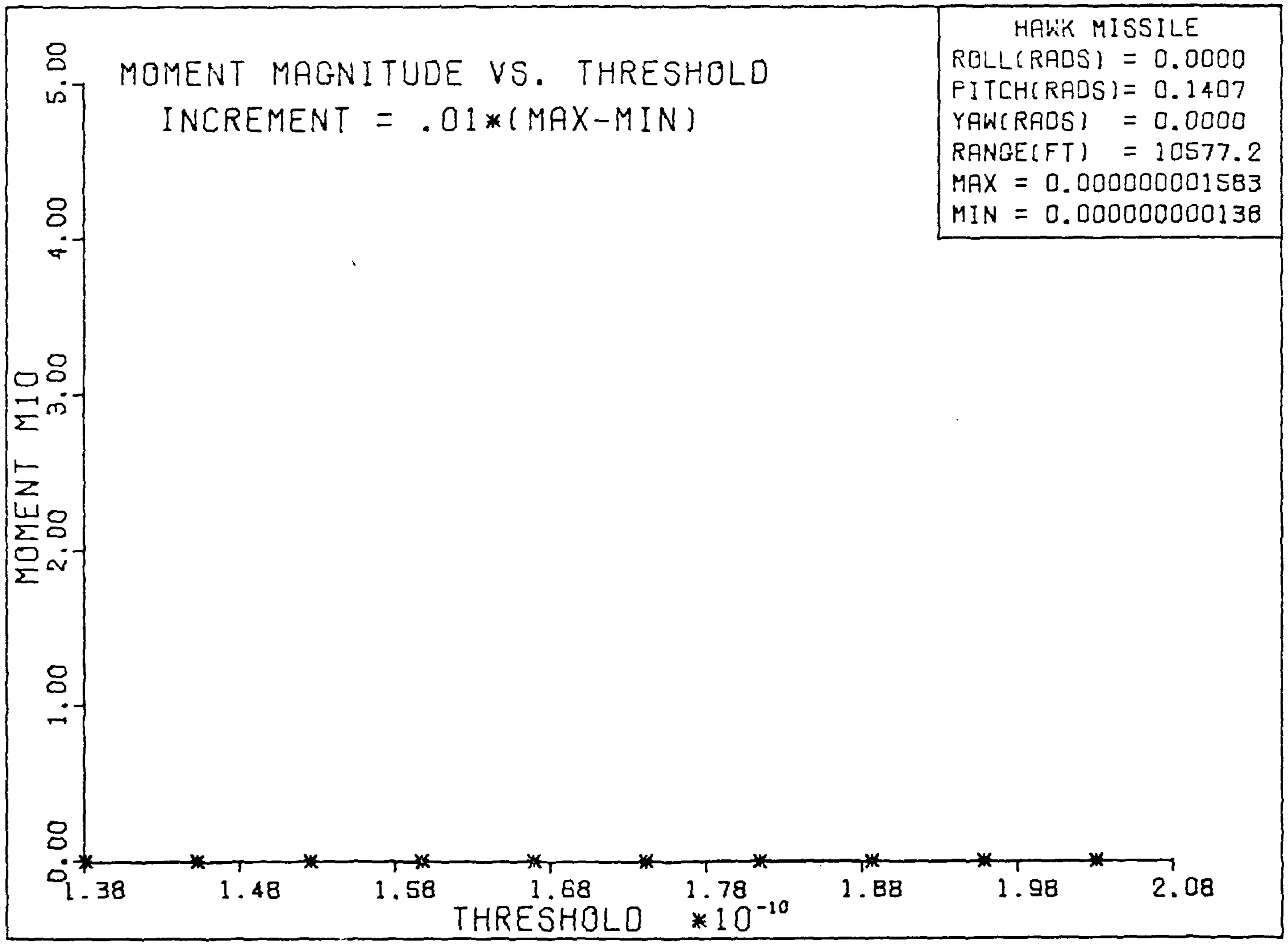

Figure C-13. Variation of Central Moment μ_{10} versus Threshold.

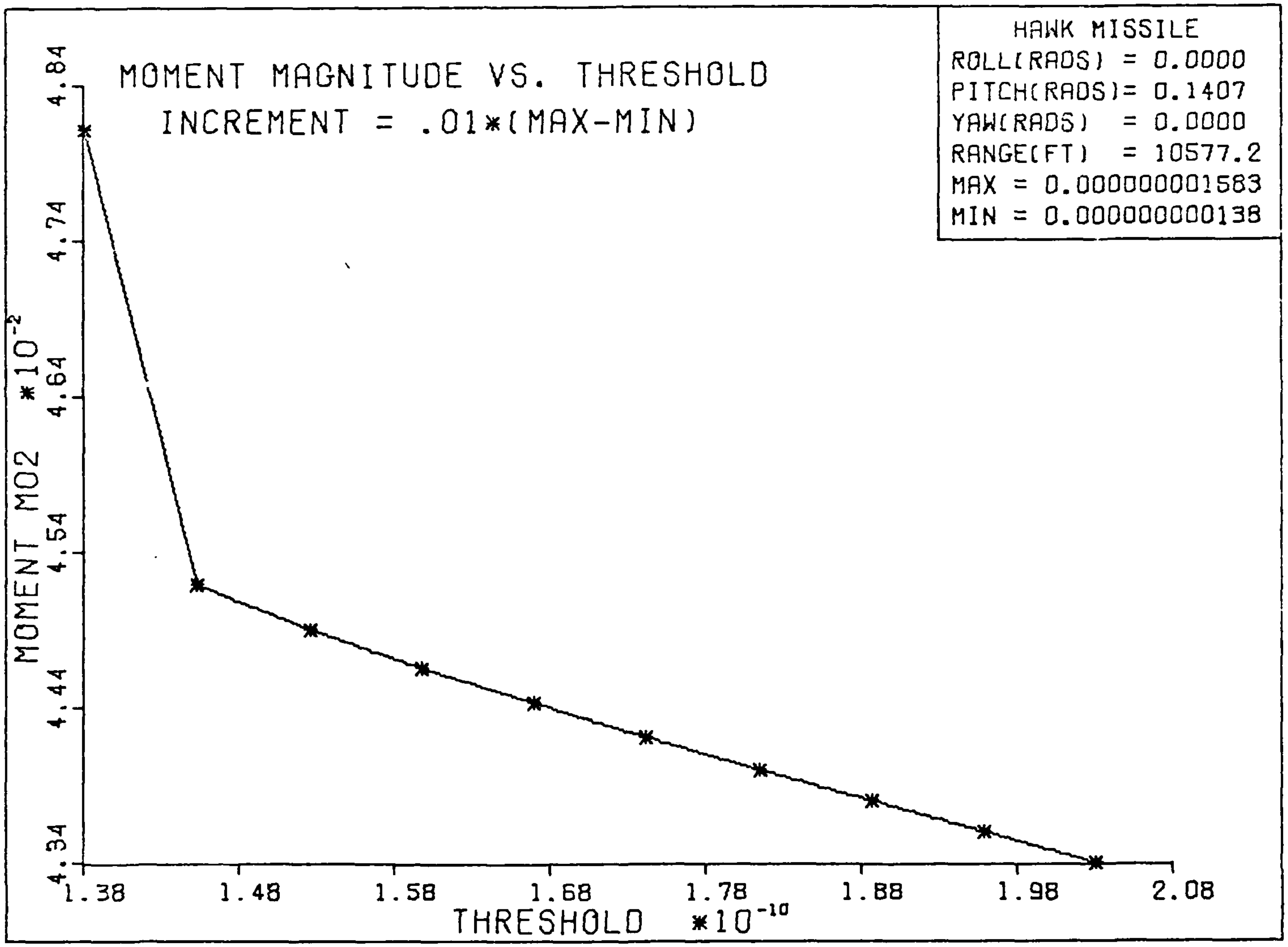

Figure C-14. Variation of Central Moment μ_{02} versus Threshold.

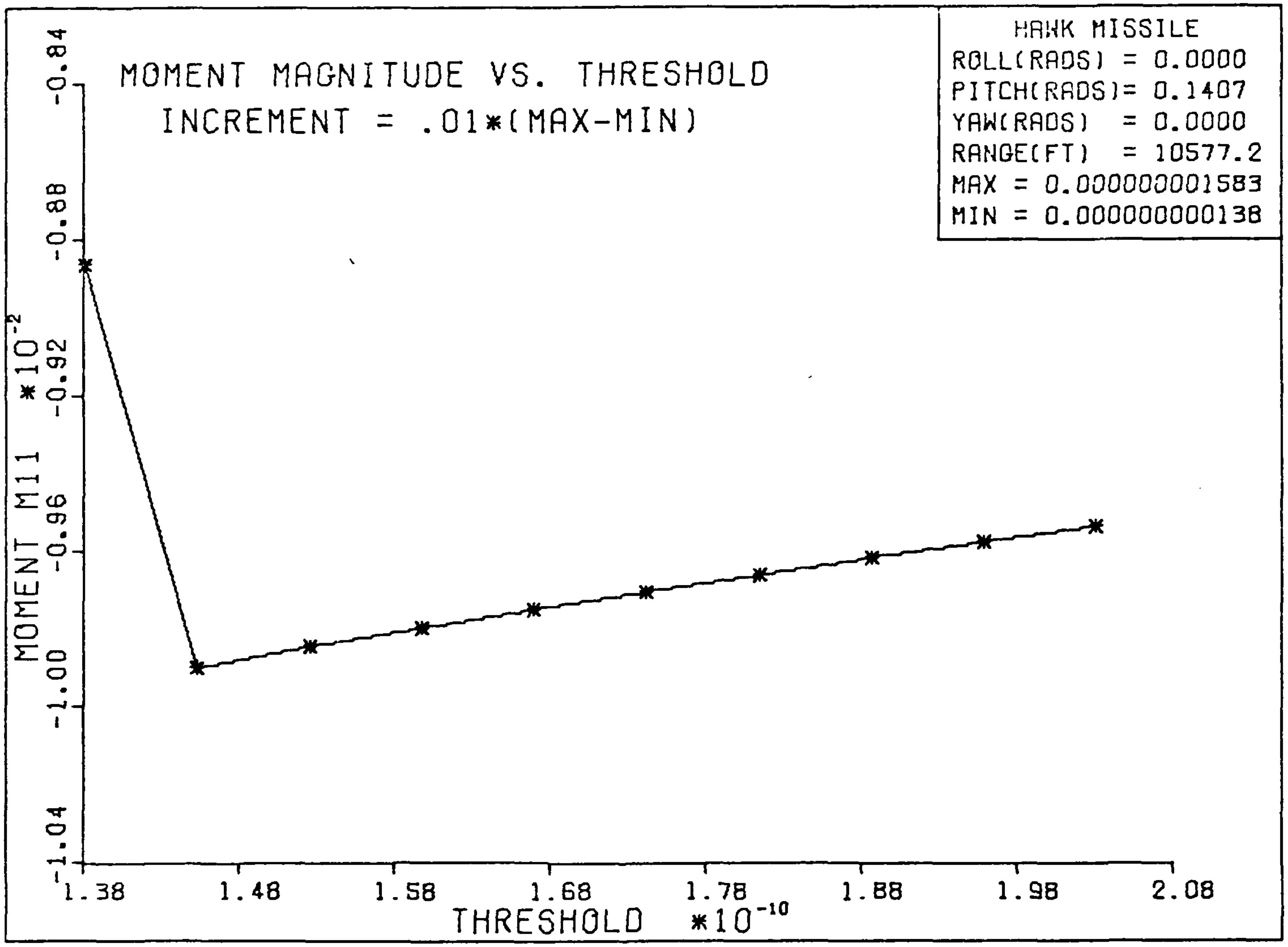

Figure C-15. Variation of Central Moment μ_{11} versus Threshold.

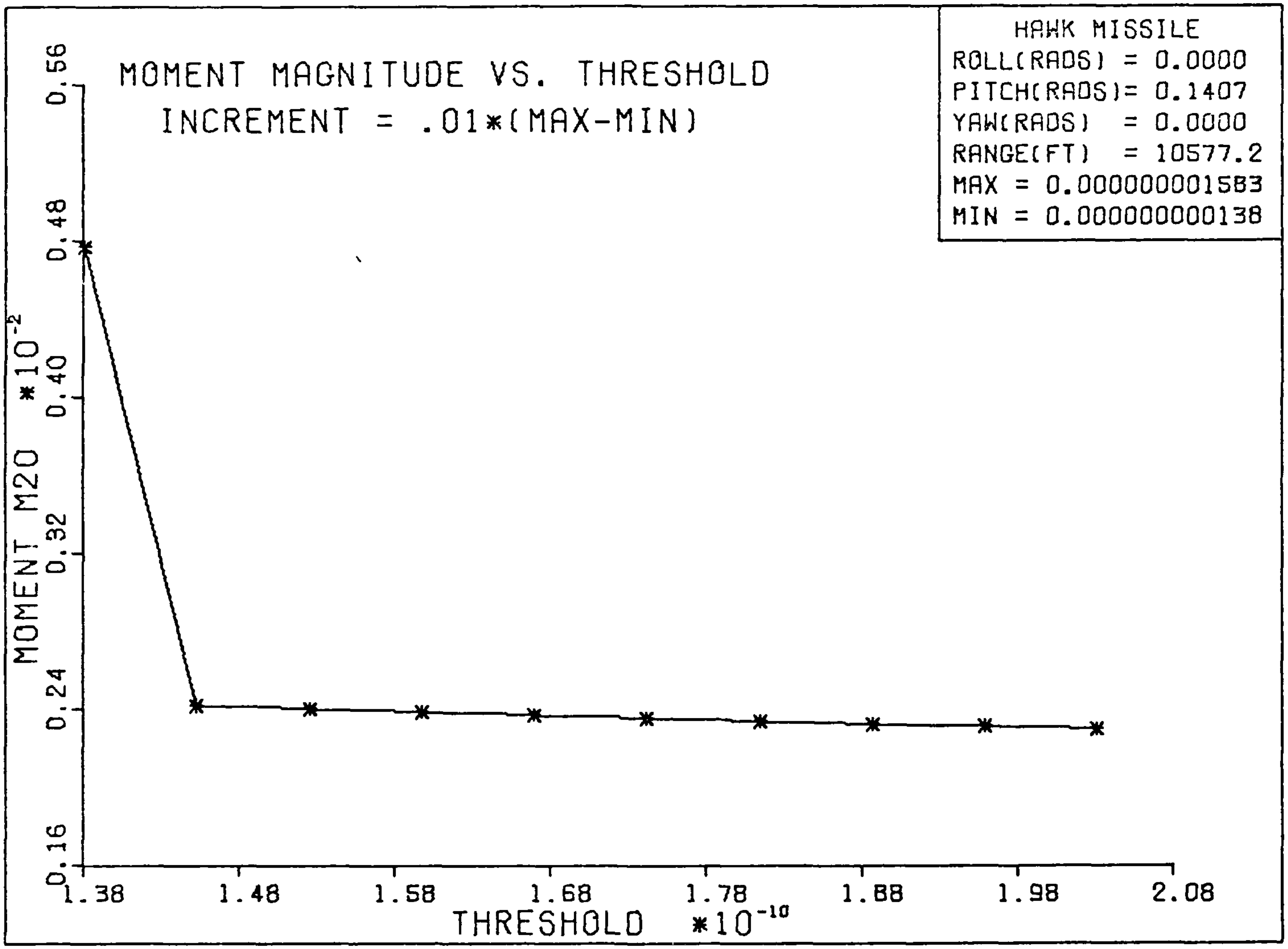

Figure C-16. Variation of Central Moment μ_{20} versus Threshold.

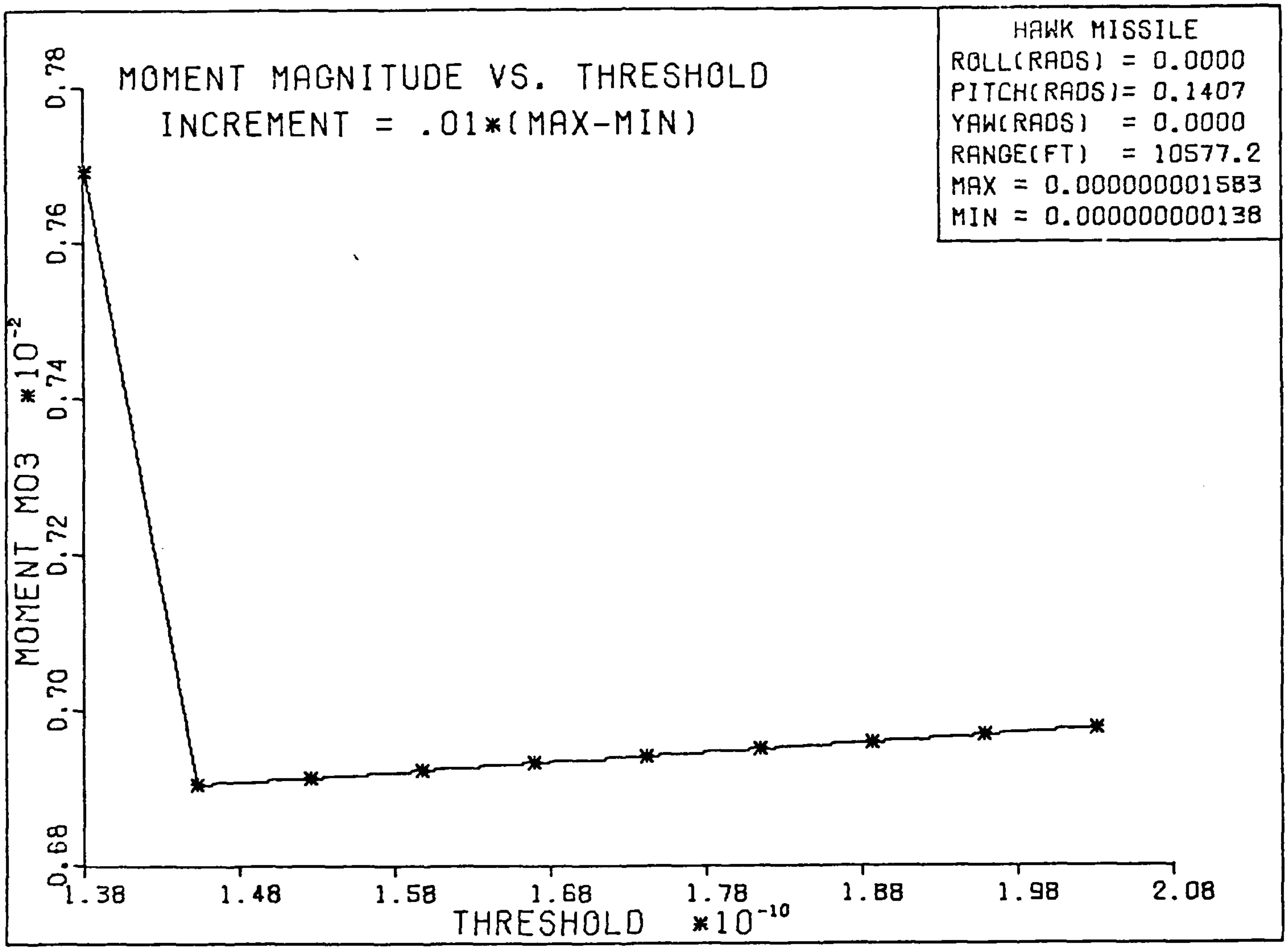

Figure C-17. Variation of Central Moment μ_{03} versus Threshold.

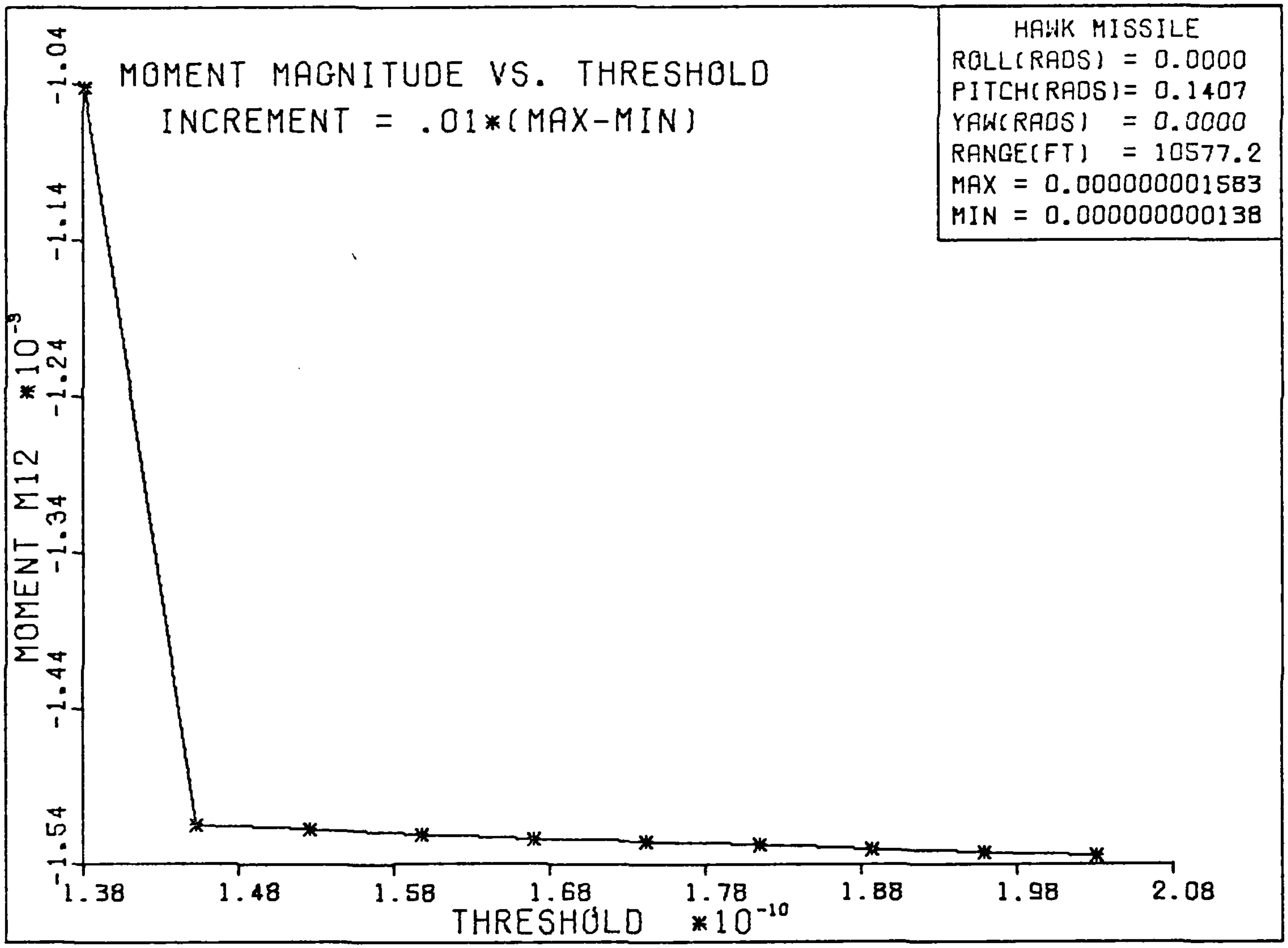

Figure C-18. Variation of Central Moment μ_{12} versus Threshold.

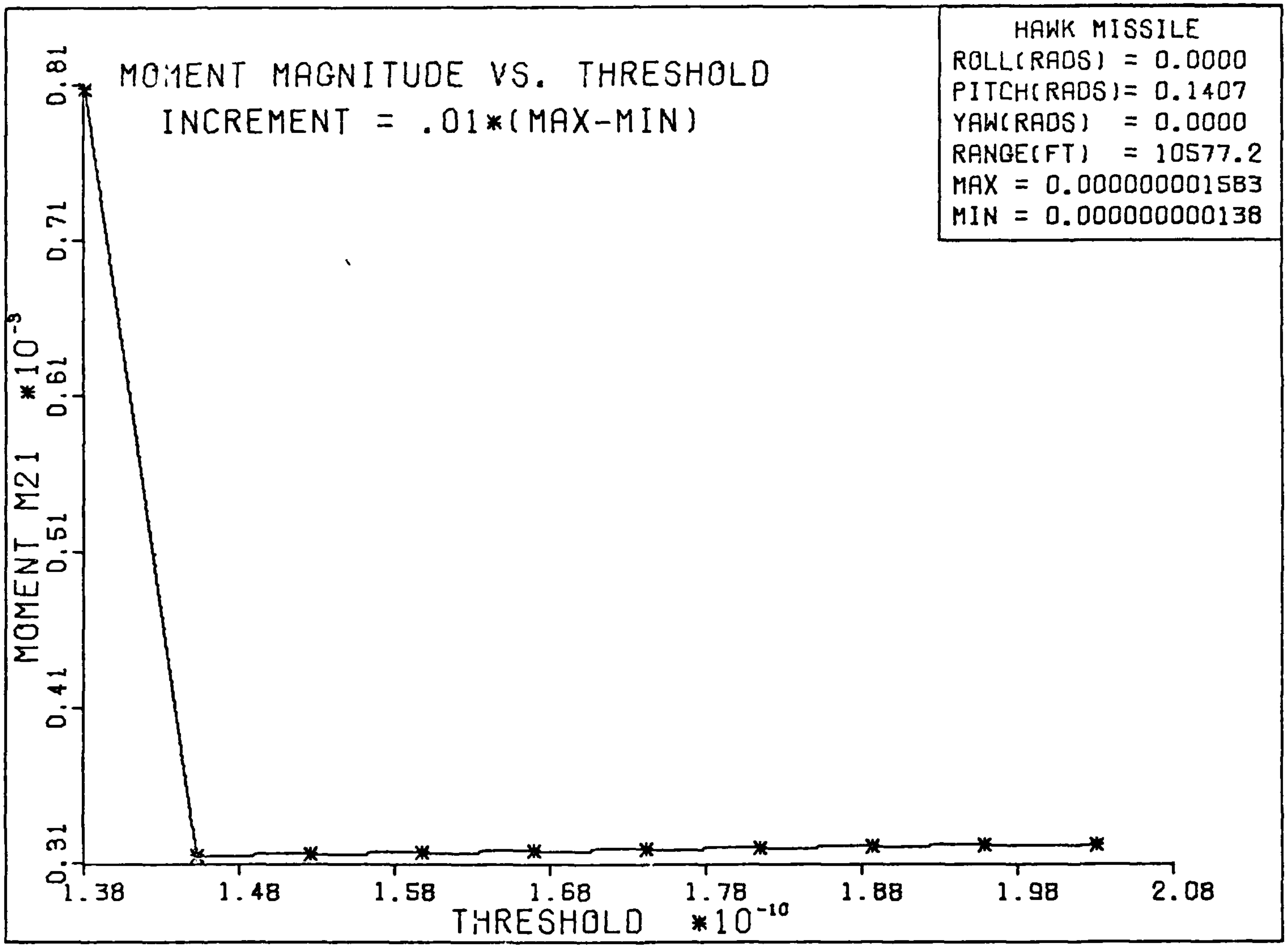

Figure C-19. Variation of Central Moment μ_{21} versus Threshold.

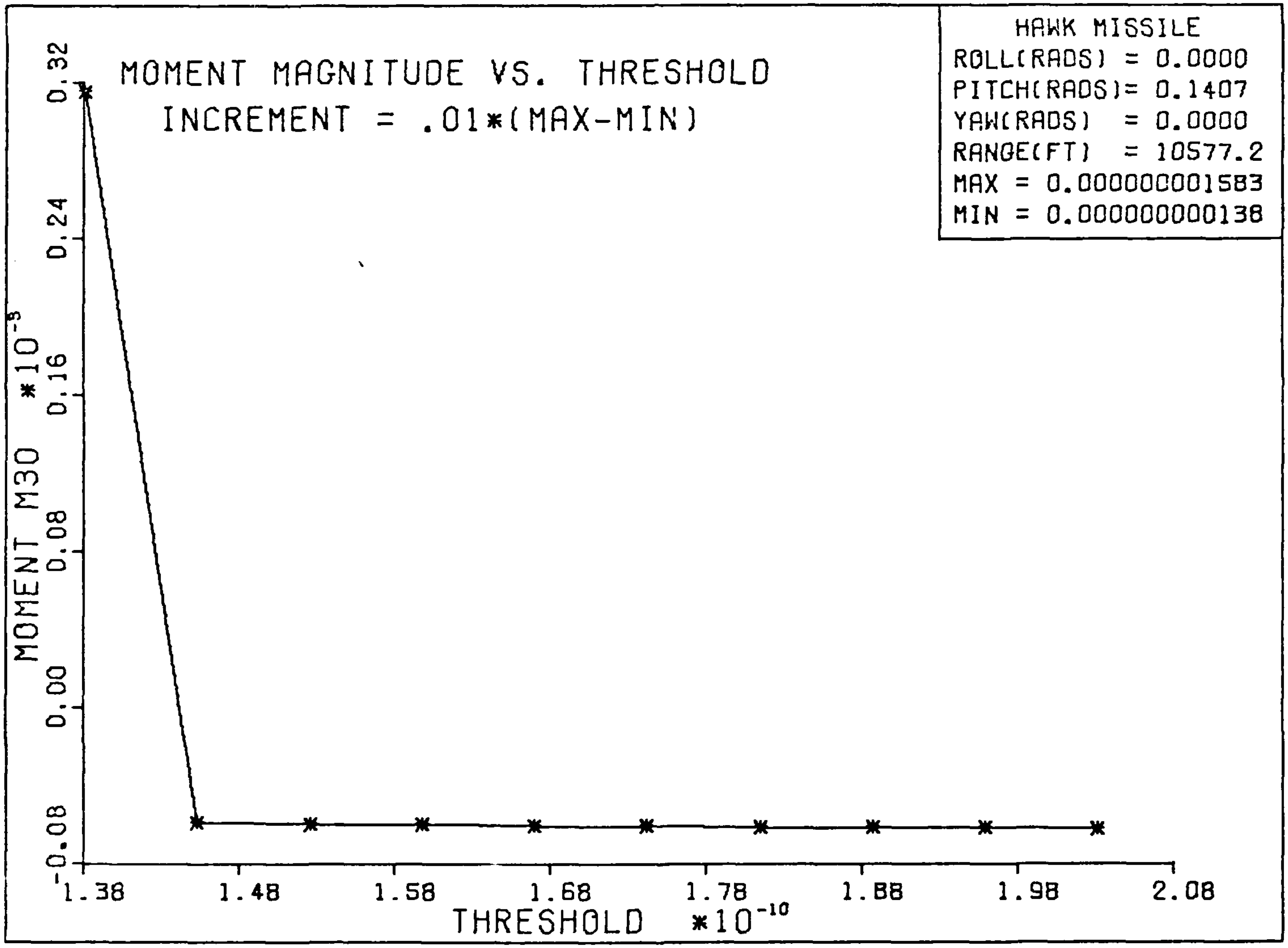

Figure C-20. Variation of Central Moment μ_{30} versus Threshold.

Appendix D

Orthogonality Relations of Group Representations

Schur's Lemmas

I. If D and D' are two irreducible representations of a Group G, having different dimensions, then if the matrix $\underline{A}$ satisfies $\underline{D}(R)\ \underline{A} = \underline{A}\ \underline{D}'(R)$ for all R in G, it follows that $\underline{A}$ is the zero matrix.

II. If the matrices $\underline{D}(R)$ are an irreducible representation of a group G, and if $\underline{A}\ \underline{D}(R) = \underline{D}(R)\underline{A}$ for all R in G, then the matrix $\underline{A}$ is a multiple of the identity matrix.

Given an irreducible representation of degree η for the group G of order g, matrix $\underline{A}$ is constructed to satisfy the conditions of Lemma II,

$$\underline{A} = \sum_{S} \underline{D}(S)\ \underline{X}\ D(S^{-1}) \tag{D-1}$$

where $\underline{X}$ is an arbitrary matrix and the summation is taken over the entire group. Then

$$\underline{D}(R)\ \underline{A} = \sum_S \underline{D}(R)\ \underline{D}(S)\ \underline{X}\ \underline{D}(S^{-1}) \tag{D-2}$$

$$= \sum_S \underline{D}(R)\ D(S)\ \underline{X}\ \underline{D}(S^{-1})\ \underline{D}(R^{-1})\ \underline{D}(R) \tag{D-3}$$

$$= \sum_S \underline{D}(RS)\ \underline{X}\ D(\left|RS\right|^{-1}) \cdot D(R) \tag{D-4}$$

$$= \underline{A}\ D(R). \tag{D-5}$$

According to the lemma, $\underline{A}$ is a multiple of the unit matrix, $\underline{A} = \lambda \times \underline{1}$. $\underline{X}$ is chosen to have all its elements zero except $X_{lm} = 1$. The constant is then denoted by λ_{lm} and

$$\sum_S D_{il}(S)\ D_{mj}(S^{-1}) = \lambda_{lm}\ \delta_{ij}\ . \tag{D-6}$$

If $\underline{D}$ is unitary,

$$\sum_S D_{il}(S)\ D^{*}_{jm}(S) = \lambda_{lm}\delta_{ij}\ . \tag{D-7}$$

To evaluate λ_{lm}, set i=j and sum over i,

$$\sum_S \sum_i D_{il}(S)\ D_{mi}(S^{-1}) = \eta\lambda_{lm}$$

$$= \sum_{S} D_{ml}(SS^{-1})$$

$$= \sum_{S} D_{ml}(E)$$

$$= \sum_{S} \delta_{ml}$$

$$= g\ \delta_{ml}$$

$$\lambda_{lm} = \frac{g}{\eta}\ \delta_{lm} \qquad \text{(D-8)}$$

Therefore,

$$\sum_{S} D_{il}(S)\ D_{mj}(S^{-1}) = \frac{g}{\eta}\ \delta_{lm}\ \delta_{ij}, \qquad \text{(D-9)}$$

and for D unitary,

$$\sum_{S} D_{il}(S)\ D_{jm}^{*}(S) = \frac{g}{\eta}\ \delta_{lm}\ \delta_{ij}, \qquad \text{(D-10)}$$

Likewise, given any two nonequivalent representations, $\underline{D}^{(1)}$ and $\underline{D}^{(2)}$, of a group G, matrix $\underline{A}$ is constructed to satisfy Lemma I,

$$A = \sum_{S} \underline{D}^{(2)}(S)\ \underline{X}\ D^{(1)}(S^{-1}), \qquad \text{(D-11)}$$

where $\underline{X}$ is an arbitrary matrix and the summation is over G.

Then

$$\underline{D}^{(2)}(R)\ \underline{A} = \sum_S \underline{D}^{(2)}(S)\ \underline{X}\ D^{(1)}(S^{-1}) \tag{D-12}$$

$$= \sum_S \underline{D}^{(2)}(R)\ \underline{D}^{(2)}(S)\ \underline{X}\ \underline{D}^{(1)}(S^{-1})$$

$$\underline{D}^{(1)}(R^{-1})\ \underline{D}^{(1)}(R) \tag{D-13}$$

$$= \sum_S \underline{D}^{(2)}(RS)\ \underline{X}\ \underline{D}^{(1)}(|RS|^{-1})\ D^{(1)}(R) \tag{D-14}$$

$$= A\ D^{(1)}(R). \tag{D-15}$$

According to Lemma I, $\underline{A}$ is the zero matrix. Thus,

$$\sum_S \underline{D}^{(2)}(S)\ \underline{X}\ \underline{D}^{(1)}(S^{-1}) = \underline{0}. \tag{D-16}$$

If $\underline{X}$ is chosen as before,

$$\sum_S D_{il}^{(2)}(S)\ D_{mj}^{(1)}(S^{-1}) = 0 \tag{D-17}$$

for all i, j, l, m. If both representations are unitary,

$$\sum_S D_{il}^{(2)}(S)\ D_{jm}^{(1)*}(S) = 0. \tag{D-18}$$

Taken together, Eqs (D-9) and (D-17) imply that for all nonequivalent irreducible representations of G,

$$\sum_R D_{il}^{(\mu)}(R)\ D_{mj}^{(\nu)}(R^{-1}) = \frac{g}{\eta_\nu}\ \delta_{\mu\nu}\ \delta_{ij}\ \delta_{lm}. \quad \text{(D-19)}$$

For the unitary case,

$$\sum_R D_{il}^{(\mu)}(R)\ D_{jm}^{(\nu)*}(R^{-1}) = \frac{g}{\eta_\mu}\ \delta_{\mu\nu}\ \delta_{ij}\ \delta_{lm}. \quad \text{(D-20)}$$

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Appendix E

Derivation of the Projection Operator

Multiply Eq (65), $O_R \Psi_i^{(\nu)} = \sum_j \Psi_j^{(\nu)} D_{ji}^{(\nu)}(R)$,

by $D_{lm}^{(\mu)*}(R)$ and sum over the entire group,

$$\sum_R D_{lm}^{(\mu)*}(R)\ O_R \Psi_i^{(\nu)} = \sum_j \Psi_i^{(\nu)} \sum_R D_{lm}^{(\mu)*}(R)\ D_{ji}^{(\nu)}(R). \qquad \text{(E-1)}$$

Due to orthogonality,

$$\sum_R D_{lm}^{(\mu)*}(R)\ O_R \Psi_i^{(\nu)} = \frac{g}{\eta_\nu} \sum_j \Psi_j^{(\nu)} \delta_{lj}\ \delta_{mi}\ \delta_{\mu\nu} \qquad \text{(E-2)}$$

$$= \frac{g}{\eta_\nu} \Psi_l^{(\nu)}\ \delta_{li}\ \delta_{\mu\nu}. \qquad \text{(E-3)}$$

Then for m=l and $\mu = \nu$,

$$\sum_R D_{ll}^{(\nu)*}(R)\ O_R\ \Psi_i^{(\nu)} = \frac{g}{\eta_\nu}\ \Psi_l^{(\nu)}\ \delta_{li} \qquad \text{(E-4)}$$

and for l - i,

$$\sum_R D_{ii}^{(\nu)*}(R)\ O_R\ \Psi_i^{(\nu)} = \frac{g}{\eta_\nu}\ \Psi_i^{(\nu)}\ . \qquad \text{(E-5)}$$

This is a necessary condition on $\Psi_i^{(\nu)}$, and in fact, it is also a sufficient condition such that Eq (65) is satisfied. This is proved by substituting Eq (E-3) into Eq (65):

$$\sum_j \Psi_j^{(\nu)} D_{ji}^{(\nu)}(S) = \frac{n_\nu}{g} \sum_j \left[\sum_R D_{jk}^{(\nu)*}(R)\; O_R \Psi_K^{(\nu)}\right] D_{ji}^{(\nu)}(S) \qquad \text{(E-6)}$$

$$= \frac{n_\nu}{g} \sum_R \left[\sum_j D_{ji}^{(\nu)}(S)\; D_{jk}^{(\nu)*}(R)\right] O_R \Psi_K^{(\nu)} \qquad \text{(E-7)}$$

but D_{ji} is Hermitian

$$= \frac{n_\nu}{g} \sum_R \left[\sum_j D_{ij}^{(\nu)*}(S^{-1})\; D_{jk}^{(\nu)*}(R)\right] O_R \Psi_K^{(\nu)} \qquad \text{(E-8)}$$

$D(AB) = D(A)\; D(B)$

$$= \frac{n_\nu}{g} \sum_R \left[D_{ik}^{(\nu)*}(S^{-1}R)\right] O_R \Psi_K^{(\nu)} \qquad \text{(E-9)}$$

$$= \frac{n_\nu}{g}\; O_S \sum_R O_S^{-1}\left[D_{ik}^{(\nu)*}(S^{-1}R)\right] O_R \Psi_K^{(\nu)} \qquad \text{(E-10)}$$

and by Eq (43)

$$= \frac{n_\nu}{g}\; O_S \sum_R D_{ik}^{(\nu)*}(R)\; O_R \Psi_K^{(\nu)} \qquad \text{(E-11)}$$

but this is Eq (E-3)

$$= O_S\; \Psi_i^{(\nu)} \qquad \text{(E-12)}$$

and the desired result is obtained.

From Eq (E-3) for m - 1,

$$\sum_R D_{11}^{(\mu)*}(R)\ O_R \psi_i^{(\nu)} = \frac{g}{\eta_\nu}\ \psi_1^{(\nu)}\ \delta_{1i}\ \delta_{\mu\nu} \qquad \text{(E-13)}$$

Define a <u>projection operator</u> to be

$$P_i^{(\mu)} = \frac{\eta_\mu}{g} \sum_R D_{ii}^{(\mu)*}(R)\ O_R \qquad \text{(E-14)}$$

such that

$$P_i^{(\mu)}\ \psi_j^{(\nu)} = \psi_i^{(\mu)}\delta_{\mu\nu}\ \delta_{ij}. \qquad \text{(E-15)}$$

If the projection operator is applied to Eq (46),

$$P_j^{(\mu)}\psi = \sum_\nu \sum_{i=1}^{\eta_\nu} P_j^{(\mu)}\ \psi_i^{(\nu)} \qquad \text{(E-16)}$$

$$= \sum_\nu \sum_{i=1}^{\eta_\nu} \psi_j^{(\mu)}\delta_{\mu\nu}\ \delta_{ij}. \qquad \text{(E-17)}$$

and due to orthogonality

$$\frac{\eta_\nu}{g} \sum_R D_{jj}^{(\mu)*}(R)\ O_R = \psi_j^{(\mu)} \qquad \text{(E-18)}$$

or

$$\psi_j^{(\mu)} = P_j^{(\mu)}\psi\ . \qquad \text{(E-19)}$$

Appendix F

Reducible Representation of the Rotation Group

$$C_n \equiv \cos n\theta$$
$$S_n \equiv \sin n\theta \tag{F-1}$$

$$\mu'_{pq} = E \quad (xC + yS)^p \quad (-xS + yC)^q \tag{F-2}$$

$$\mu'_{00} = \mu_{00} \tag{F-3}$$

$$\begin{bmatrix} \mu'_{01} \\ \mu'_{10} \end{bmatrix} = \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{bmatrix} \mu_{01} \\ \mu_{10} \end{bmatrix} \tag{F-4}$$

$$\begin{bmatrix} \mu'_{02} \\ \mu'_{11} \\ \mu'_{20} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} C_2 + 1 & -2S_2 & -(C_2-1) \\ S_2 & 2C_2 & -S_2 \\ -(C_2-1) & 2S_2 & C_2+1 \end{bmatrix} \begin{bmatrix} \mu_{02} \\ \mu_{11} \\ \mu_{20} \end{bmatrix} \tag{F-5}$$

$$\begin{bmatrix} \mu'_{03} \\ \mu'_{12} \\ \mu'_{12} \\ \mu'_{30} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} C_3+3C & -3(S_3+S) & -3(C_3-C) & S_3-3S \\ S_3+S & 3C_3+C & -(3S_3-S) & -(C_3-C) \\ -(C_3-C) & 3S_3-S & 3C_3+C & -(S_3+S) \\ -(S_3-3S) & -3(C_3-C) & 3(S_3+S) & C_3+3C \end{bmatrix} \begin{bmatrix} \mu_{03} \\ \mu_{12} \\ \mu_{21} \\ \mu_{30} \end{bmatrix} \tag{F-6}$$

$$\begin{bmatrix} \mu'_{04} \\ \mu'_{13} \\ \mu'_{22} \\ \mu'_{31} \\ \mu'_{40} \end{bmatrix} = \frac{1}{8} \begin{bmatrix} C_4+4C_2+3 & -4(S_4+2S_2) & -6(C_4-1 & 4(S_4-2S_2) & C_4-4C_2+3 \\ S_4+2S_2 & 4(C_4+C_2) & -6S_4 & -4(C_4-C_2) & S_4-2S_2 \\ -(C_4-1) & 4S_4 & 2(3S_4+1) & -4S_4 & -(C_4-1) \\ -(S_4-2S_2) & -4(C_4-C_2) & 6S_4 & 4(C_4+C_2) & -(S_4+2S_2) \\ C_4-4C_2+3 & -4(S_4-2S_2) & -6(C_4-1) & 4(S_4+2S_2) & C_4+4C_2+3 \end{bmatrix} \begin{bmatrix} \mu_{04} \\ \mu_{13} \\ \mu_{22} \\ \mu_{31} \\ \mu_{40} \end{bmatrix} \qquad \text{(F-7)}$$

$$
\begin{bmatrix} \mu'_{05} \\ \mu'_{14} \\ \mu'_{23} \\ \mu'_{32} \\ \mu'_{41} \\ \mu'_{50} \end{bmatrix} = \frac{1}{16}
\begin{bmatrix}
C_5+5C_3+10C & -5(S_5+3S_3+2S & -10(C_5+C_3-2C) & 10(S_5-S_3-2S) & 5(C_5-3C_3+2C) & -(S_5+5S_3+10S \\
S_5+3S_3+2S & 5C_5+9C_3+2C & -2(5S_5+3S_3-2S) & -2(5C_5-3C_3-2C) & 5S_5+9S_3+2S & C_5-3C_3+2C \\
-(C_5+C_3-2C) & 5S_5+3S_3-2S & 2(5C_5+C_3+2C) & -2(5S_5-S_3+2S) & -(5C_5-3C_3-2C) & S_5-S_3-2S \\
-(S_5-S_3-2S) & -(5C_5-3C_3-2C) & 2(5S_5-S_3+2S) & 2(5C_5+C_3+2C) & -(5S_5+3S_3-2S) & -(C_5+C_3-2C) \\
C_5-3C_3-2C & -(5S_5-9S_3+2S) & -2(5C_5-3C_3-2C) & 2(5S_5+3S_3-2S) & 5C_5+9C_3+2C & -(S_5+3S_3+2S) \\
S_5-5S_3+10S & 5(C_5-3C_3+2C) & -10(S_5-S_3-2S) & -10(C_5+C_3-2C) & 5(S_5+3S_3+2S) & C_5+5C_3+10C
\end{bmatrix}
\begin{bmatrix} \mu_{05} \\ \mu_{14} \\ \mu_{23} \\ \mu_{32} \\ \mu_{41} \\ \mu_{50} \end{bmatrix}
\qquad \text{(F-8)}
$$

$$
\begin{bmatrix} \mu'_{06} \\ \mu'_{15} \\ \mu'_{24} \\ \mu'_{33} \\ \mu'_{42} \\ \mu'_{51} \\ \mu'_{60} \end{bmatrix}
= \frac{1}{32}
\left[\begin{array}{ll}
C_6+6C_4+15C_2+10 & -6(S_6+4S_4+5S_2) \\
S_6+4S_4+5S_2 & 2(3C_6+8C_4+5C_2) \\
-(C_6+2C_4-C_2-2) & 2(3S_6+4S_4-S_2) \\
-(S_6-3S_2) & -6(C_6-C_2) \\
C_6-2C_4-C_2+2 & -2(3S_6-4S_4-S_2) \\
5S_6-4S_4+5S_2 & 2(3C_6-8C_4-5C_2) \\
-(C_6-6C_4+15C_2-10) & 6(S_6-4S_4+5S_2)
\end{array}\right.
$$

$$
\begin{array}{lll}
-15(C_6+2C_4-C_2-2) & 20(S_6-3S_2) & 15(C_6-2C_4-C_2+2) \\
-5(3S_6+4S_4-S_2) & -20(C_6-C_2) & 5(3S_6-4S_4-S_2) \\
15C_6+10C_4+C_2+6 & -4(5S_6+S_2) & -(15C_6-10C_4+C_2-6) \\
3(5S_6+S_2) & 4(5C_6+3C_2) & -3(5S_6+S_2) \\
-(15C_6-10C_4+C_2-6) & 4(5S_6+S_2) & 15C_6+10C_4+C_2+6 \\
-5(3S_6-4S_4-S_2) & -20(C_7-C_2) & 5(3S_6+4S_4-S_2) \\
15(C_6-2C_4-C_2+2) & -20(S_6-3S_2) & -15(C_6+2C_4-C_2-2)
\end{array}
$$

$$
\left.\begin{array}{ll}
-6(S_6-4S_4+5S_2) & -(C_6-6C_4+5C_2+10) \\
2(3C_6-8C_4+5C_2) & -(S_6-4S_4+5S_2) \\
2(3S_6-4S_4-S_2) & C_6-2C_4-C_2+2 \\
-6(C_6-C_2) & S_6-3S_2 \\
-2(3S_6+4S_4-S_2) & -(C_6+2C_4-C_2-2)
\end{array}\right]
\begin{bmatrix} \mu_{06} \\ \mu_{16} \\ \mu_{24} \\ \mu_{33} \\ \mu_{42} \end{bmatrix}
$$

$$\left.\begin{matrix} 2(3C_6+8C_4+5C_2) & -(S_6+4S_4+5S_2) \\ 6(S_6+4S_4+5S_2) & C_6+6C_4+15C_2+10 \end{matrix}\right] \begin{bmatrix} \mu_{51} \\ \mu_{60} \end{bmatrix} \tag{F-9}$$

Appendix G

Useful Trigonometric Identities

$$C_n \equiv \cos n\theta$$

$$S_n \equiv \sin n\theta \qquad \text{(G-1)}$$

$$S_A S_B = \frac{1}{2}\left(C_{A-B} - C_{A+B}\right)$$

$$C_A C_B = \frac{1}{2}\left(C_{A+B} + C_{A-B}\right)$$

$$S_A C_B = \frac{1}{2}\left(S_{A+B} + S_{A-B}\right) \qquad \text{(G-2)}$$

$$S^2 = \frac{1}{2}\left(-C_2 + 1\right)$$

$$CS = \frac{1}{2}\ S_2$$

$$C^2 = \frac{1}{2}\left(C_2 + 1\right) \qquad \text{(G-3)}$$

$$S^3 = \frac{1}{4}\left(-S_3 + 3S\right)$$

$$CS^2 = \frac{1}{4}\left(-C_3 + C\right)$$

$$C^2S = \frac{1}{4}\left(S_3 + S\right)$$

$$c^3 = \frac{1}{4}\left(c_3 + 3c\right) \tag{G-4}$$

$$s^4 = \frac{1}{8}\left(c_4 - 4c_2 + \frac{6}{2}\right)$$

$$cs^3 = \frac{1}{8}\left(-s_4 + 2s_2\right)$$

$$c^2s^2 = \frac{1}{8}\left(-c_4 + 1\right)$$

$$c^3s = \frac{1}{8}\left(s_4 + 2s_2\right)$$

$$c^4 = \frac{1}{8}\left(c_4 + 4c_2 + \frac{6}{2}\right) \tag{G-5}$$

$$s^5 = \frac{1}{16}\left(s_5 - 5s_3 + 10s\right)$$

$$cs^4 = \frac{1}{16}\left(c_5 - 3c_3 + 2c\right)$$

$$c^2s^3 = \frac{1}{16}\left(-s_5 + s_3 + 2s\right)$$

$$c^3s^2 = \frac{1}{16}\left(-c_5 - c_3 + 2c\right)$$

$$c^4s = \frac{1}{16}\left(s_5 + 3s_3 + 2s\right)$$

$$c^5 = \frac{1}{16}\left(c_5 + 5c_3 + 10c\right) \tag{G-6}$$

$$s^6 = \frac{1}{32}\left(-c_6 + 6c_4 - 15c_2 + \frac{20}{2}\right)$$

$$cs^5 = \frac{1}{32}\left(s_6 - 4s_4 + 5s_2\right)$$

$$c^2s^4 = \frac{1}{32}\left(c_6 - 2c_4 - c2 + 2\right)$$

$$c^3s^3 = \frac{1}{32}\left(-s_6 + 3s_2\right)$$

$$c^4s^2 = \frac{1}{32}\left(-c_6 = 2c_4 + c_2 + 2\right)$$

$$c^5s = \frac{1}{32}\left(s_6 + 4s_4 + 5s_2\right)$$

$$c^6 = \frac{1}{32}\left(c_6 + 6c_4 + 15c_2 + \frac{20}{2}\right) \tag{G-7}$$

Appendix H

Projected Moment Vectors

$$X_{00}^{(0)} = \mu_{00} \qquad Y_{00}^{(0)} = 0 \tag{H-1}$$

$$X_{01}^{(1)} = \mu_{01} \qquad Y_{01}^{(1)} = \mu_{10} \tag{H-2}$$

$$X_{02}^{(0)} = \mu_{20}+\mu_{02} \qquad Y_{02}^{(0)} = 0 \tag{H-3}$$

$$X_{02}^{(2)} = -\frac{1}{2}\ (\mu_{20}-\mu_{02}) \qquad Y_{02}^{(2)} = -\mu_{11} \tag{H-4}$$

$$X_{03}^{(1)} = \frac{3}{4}\ (\mu_{21}+\mu_{03}) \qquad Y_{03}^{(1)} = \frac{3}{4}\ (\mu_{30}+\mu_{12}) \tag{H-5}$$

$$X_{03}^{(3)} = -\frac{1}{4}\ (3\mu_{21}-\mu_{03}) \qquad Y_{03}^{(3)} = \frac{1}{4}\ (\mu_{30}-3\mu_{12}) \tag{H-6}$$

$$X_{12}^{(1)} = \frac{1}{4}\ (\mu_{30}+\mu_{12}) \qquad Y_{12}^{(1)} = \frac{1}{4}\ (\mu_{21}+\mu_{03}) \tag{H-7}$$

$$X_{04}^{(0)} = \frac{3}{4}\ (\mu_{20}+2\mu_{22}+\mu_{04}) \qquad Y_{04}^{(0)} = 0 \tag{H-8}$$

$$X_{04}^{(2)} = -\frac{1}{2}\ (\mu_{40}\ -\mu_{04}) \qquad Y_{04}^{(2)} = -(\mu_{31}+\mu_{13}) \tag{H-9}$$

$$X_{04}^{(4)} = \frac{1}{8}\ (\mu_{40}-6\mu_{22}+\mu_{04}) \qquad Y_{04}^{(4)} = \frac{1}{2}\ (\mu_{31}-\mu_{13}) \tag{H-10}$$

$$X_{13}^{(2)} = \frac{1}{2}\ (\mu_{31}+\mu_{13}) \qquad Y_{13}^{(2)} = -\frac{1}{4}\ (\mu_{40}-\mu_{04}) \tag{H-11}$$

$$X_{05}^{(1)} = \frac{5}{8} (\mu_{41}+2\mu_{23} +\mu_{05})$$

$$Y_{05}^{(1)} = \frac{5}{8} (\mu_{50}+2\mu_{32}+\mu_{14}) \tag{H-12}$$

$$X_{05}^{(3)} = -\frac{5}{16} (3\mu_{41}+2\mu_{23}-\mu_{05})$$

$$Y_{05}^{(3)} = \frac{5}{16} (\mu_{50}-2\mu_{32}-3\mu_{14}) \tag{H-13}$$

$$X_{05}^{(5)} = \frac{1}{16} (5\mu_{41}-10\mu_{23}+\mu_{05})$$

$$Y_{05}^{(5)} = -\frac{1}{16} (\mu_{50}-10\mu_{32}+5\mu_{14}) \tag{H-14}$$

$$X_{14}^{(1)} = \frac{1}{8} (\mu_{50}+2\mu_{32}+\mu_{14})$$

$$Y_{14}^{(1)} = \frac{1}{8} (\mu_{41}+2\mu_{23}+\mu_{05}) \tag{H-15}$$

$$X_{14}^{(3)} = -\frac{3}{16} (\mu_{50}-2\mu_{32}-3\mu_{14})$$

$$Y_{14}^{(3)} = \frac{3}{-16} (3\mu_{41}+2\mu_{23}-\mu_{15}) \tag{H-16}$$

$$X_{06}^{(0)} = \frac{20}{32} (\mu_{60}+3\mu_{42}+3\mu_{24}+\mu_{06})$$

$$Y_{06}^{(0)} = 0 \tag{H-17}$$

$$X_{06}^{(2)} = \frac{15}{32} (\mu_{60}+\mu_{42}-\mu_{24}-\mu_{06})$$

$$Y_{06}^{2)} = -\frac{30}{32} (\mu_{51}+2\mu_{33}+\mu_{15}) \qquad \text{(H-18)}$$

$$X_{06}^{(4)} = \frac{6}{32} (\mu_{60}-5\mu_{42}-5\mu_{24}+\mu_{06})$$

$$Y_{06}^{(4)} = \frac{24}{32} (\mu_{51}-\mu_{15}) \qquad \text{(H-19)}$$

$$X_{06}^{(6)} = \frac{1}{32} (\mu_{60}-15\mu_{42}+15\mu_{24}-\mu_{06})$$

$$Y_{06}^{(6)} = -\frac{2}{32} (3\mu_{51}-10\mu_{33}+3\mu_{15}) \qquad \text{(H-20)}$$

$$X_{15}^{(2)} = \frac{10}{32} (\mu_{51}+2\mu_{33}+\mu_{15})$$

$$Y_{15}^{(2)} = -\frac{5}{32} (\mu_{60}+\mu_{42}-\mu_{24}-\mu_{06}) \qquad \text{(H-21)}$$

$$X_{15}^{(4)} = -\frac{16}{32} (\mu_{51}-\mu_{15})$$

$$Y_{15}^{(4)} = \frac{4}{32} (\mu_{60}-5\mu_{42}-5\mu_{24}+\mu_{06}) \qquad \text{(H-22)}$$

Appendix I

Moment Invariants From Projected Moment Vectors

$$ {}_{00}I_{00}^{(0)} = 4\mu_{00}^{2} \tag{I-1}$$

$$ {}_{00}I_{02}^{(0)} = 2\mu_{00}(\mu_{20}+\mu_{02}) \tag{I-2}$$

$$ {}_{00}I_{04}^{(0)} = \frac{3}{2}\mu_{00}\ (\mu_{40}+2\mu_{22}+\mu_{04}) \tag{I-3}$$

$$ {}_{02}I_{02}^{(0)} = (\mu_{20}+\mu_{02})^{2} \tag{I-4}$$

$$ {}_{02}I_{04}^{(0)} = \frac{3}{4}\ (\mu_{20}+\mu_{02})\ \ (\mu_{40}+2\mu_{22}+\mu_{04}) \tag{I-5}$$

$$ {}_{04}I_{04}^{(0)} = \frac{9}{16}\ (\mu_{40}+2\mu_{22}+\mu_{04})^{2} \tag{I-6}$$

$$ {}_{01}I_{01}^{(2)} = \mu_{01}^{2} + \mu_{10}^{2} \tag{I-7}$$

$$ {}_{01}I_{03}^{(1)} = \frac{3}{4}\ \mu_{01}(\mu_{21}+\mu_{03}) + \frac{3}{4}\ \mu_{10}(\mu_{30}+\mu_{12}) \tag{I-8}$$

$$ {}_{01}I_{12}^{(1)} = \frac{1}{4}\ \mu_{01}(\mu_{30}+\mu_{12}) - \frac{1}{4}\ \mu_{10}(\mu_{21}+\mu_{03}) \tag{I-9}$$

$$ {}_{01}I_{05}^{(1)} = \frac{5}{8}\ \mu_{01}(\mu_{41}+2\mu_{23}+\mu_{05}) + \frac{5}{8}\ \mu_{10}(\mu_{50}+2\mu_{32}+\mu_{14}) \tag{I-10}$$

$$ {}_{01}I_{14}^{(1)} = \frac{1}{8}\ \mu_{01}(\mu_{50}+2\mu_{32}+\mu_{14}) - \frac{1}{8}\ \mu_{10}(\mu_{41}+2\mu_{23}+\mu_{05}) \tag{I-11}$$

$$ {}_{03}I_{03}^{(1)} = \frac{9}{16} (\mu_{21}+\mu_{03})^2 + \frac{9}{16} (\mu_{30}+\mu_{12})^2 \tag{I-12} $$

$$ {}_{03}I_{05}^{(1)} = \frac{15}{32} (\mu_{21}+\mu_{03}) (\mu_{14}+2\mu_{23}+\mu_{05}) + \frac{15}{32} (\mu_{30}+\mu_{12}) (\mu_{50}+2\mu_{32}+\mu_{14}) \tag{I-13} $$

$$ {}_{03}I_{14}^{(1)} = \frac{3}{32} (\mu_{21}+\mu_{03}) (\mu_{50}+2\mu_{32}+\mu_{14}) - \frac{3}{32} (\mu_{30}+\mu_{12}) (\mu_{41}+2\mu_{23}+\mu_{05}) \tag{I-14} $$

$$ {}_{05}I_{05}^{(1)} = \frac{25}{64} (\mu_{41}+2\mu_{23}+\mu_{05})^2 + \frac{25}{64} (\mu_{50}+2\mu_{32}+\mu_{14})^2 \tag{I-15} $$

$$ {}_{02}I_{02}^{(2)} = \frac{1}{4} (\mu_{20}-\mu_{02})^2 + \mu_{11}^2 \tag{I-16} $$

$$ {}_{02}I_{04}^{(2)} = \frac{1}{4} (\mu_{20}-\mu_{02}) (\mu_{40}-\mu_{04}) + \mu_{11} (\mu_{31}+\mu_{13}) \tag{I-17} $$

$$ {}_{02}I_{13}^{(2)} = \frac{1}{4} (\mu_{20}-\mu_{02}) (\mu_{31}+\mu_{13}) + \frac{1}{4} \mu_{11} (\mu_{40}-\mu_{04}) \tag{I-18} $$

$$ {}_{04}I_{04}^{(2)} = \frac{1}{4} (\mu_{40}-\mu_{04})^2 + (\mu_{31}+\mu_{13})^2 \tag{I-19} $$

$$ {}_{03}I_{03}^{(3)} = \frac{1}{16} (3\mu_{21}-\mu_{03})^2 + \frac{1}{16} (\mu_{30}-3\mu_{12})^2 \tag{I-20} $$

$$ {}_{03}I_{05}^{(3)} = \frac{5}{64} (3\mu_{21}-\mu_{03}) (3\mu_{41}+2\mu_{23}-\mu_{05}) + \frac{5}{64} (\mu_{30}-3\mu_{12}) (\mu_{50}-2\mu_{32}-3\mu_{14}) \tag{I-21} $$

$$ {}_{03}I_{14}^{(3)} = \frac{3}{64} (3\mu_{21}-\mu_{03}) (\mu_{50}-2\mu_{32}-3\mu_{14}) $$

$$ -\frac{3}{64} (\mu_{30}-3\mu_{12}) (3\mu_{41}+2\mu_{23}-\mu_{05}) \quad \text{(I-22)} $$

$$ {}_{05}I_{05}^{(3)} = \frac{25}{256} (3\mu_{41}+2\mu_{23}-\mu_{05})^2 $$

$$ +\frac{25}{256} (\mu_{50}-2\mu_{32}-3\mu_{14})^2 \quad \text{(I-23)} $$

$$ {}_{04}I_{04}^{(4)} = \frac{1}{64} (\mu_{40}-6\mu_{22}+\mu_{04})^2 + \frac{1}{4} (\mu_{31}-\mu_{13})^2 \quad \text{(I-24)} $$

$$ {}_{05}I_{05}^{(5)} = \frac{1}{256} (5\mu_{41}-10\mu_{23}+\mu_{05})^2 $$

$$ +\frac{1}{256} (\mu_{50}-10\mu_{32}+5\mu_{14})^2 \quad \text{(I-25)} $$

Appendix J

A Complete Set of Moment Invariants Through Fourth Order Moments

$_2{}^{(0)}$

$$ {}_2I_2^{(0)} = (\mu_{20} + \mu_{02})^2 \tag{J-1} $$

$$ {}_2I_2^{(2)} = (\mu_{20} - \mu_{02})^2 + 4\mu_{11}{}^2 \tag{J-2} $$

$$ {}_3I_3^{(1)} = (\mu_{21} + \mu_{03})^2 + (\mu_{30}+\mu_{12})^2 \tag{J-3} $$

$$ {}_3I_3^{(3)} = (3\mu_{21} - \mu_{03})^2 + (3\mu_{12}-\mu_{20})^2 \tag{J-4} $$

$$ \begin{aligned} {}_3I_2^{(2)} = {} & (\mu_{02}-\mu_{20})\ (5\mu_{03}^2 + 2\mu_{21}\mu_{03} - 3\mu_{21}^2 + 3\mu_{12}^2 \\ & -2\mu_{30}\mu_{12} - 5\mu_{30}^2) + 4\mu_{11}\ (5\mu_{12}\mu_{03} + \mu_{30}\mu_{03} \\ & +9\mu_{21}\mu_{12} - 5\mu_{30}\mu_{21}) \end{aligned} \tag{J-5} $$

$$ \begin{aligned} {}_3I_2^{(6)} = {} & (\mu_{03}^2 - 6\mu_{21}\mu_{03} + 9\mu_{21}^2 - 9\mu_{12}^2 + 6\mu_{30}\mu_{12} - \mu_{30}^2) \\ & \cdot(\mu_{02}^3 - 3\mu_{02}^2\mu_{20} - 12\mu_{02}\mu_{11}^2 + 2\mu_{20}\mu_{11}^2 + 3\mu_{20}^2\ \mu_{02} - \mu_{20}^3) \\ & +4\mu_{11}(3\mu_{12}\mu_{03} + \mu_{30}\mu_{03} - 9\mu_{21}\mu_{12} - 3\mu_{30}\mu_{21}) \end{aligned} $$

$$\cdot(3\mu_{02}^2 - 4\mu_{11}^2 - 6\mu_{02}\mu_{20} + 3\mu_{20}^2) \tag{J-6}$$

$$_4I_4^{(0)} = (\mu_{40} + 2\mu_{22} + \mu_{04})^2 \tag{J-7}$$

$$_4I_4^{(2)} = (\mu_{40} - \mu_{04})^2 + 4(\mu_{31} + \mu_{13})^2 \tag{J-8}$$

$$_4I_4^{(4)} = (\mu_{40} - 6\mu_{22} + \mu_{04})^2 + 4\ (\mu_{31} - \mu_{13})^2 \tag{J-9}$$

$$_4I_2^{(2)} = (\mu_{20} - \mu_{02})\ (\mu_{40} - \mu_{04}) + 4\mu_{11}\ (\mu_{31} + \mu_{13}) \tag{J-10}$$

$$_4I_2^{(4)} = (\mu_{31} + \mu_{13})\ (\mu_{02}^2 - 2\mu_{02}\mu_{20} - 4\mu_{11}^2 + \mu_{20}^2)$$

$$+2\mu_{11}(\mu_{04} - \mu_{40})\ (\mu_{20} - \mu_{02}) \tag{J-11}$$

Appendix K

Computer Programs

Program Name: CENTER

Purpose: To calculate recursively and to normalize two-dimensional central moments from two-dimensional raw moments.

Method: The pq-th central moment (μ_{pq}) is calculated recursively from the pq-th raw moment (M_{pq}) and the lower order central moments using the formula.

$$\mu_{pq} = \frac{M_{pq}}{M_{00}} - \sum_{i=0}^{p+q-1} \sum_{j=S}^{T} \binom{p}{p-j}\binom{q}{q-i+j} \bar{x}^{\,p-j}\, \bar{y}^{\,q-i+j}\, \mu_{j,i-j}$$

where

$$S = \frac{1}{2}\left[(i-q) + |i-q|\right]$$

$$T = \frac{1}{2}\left[(p+i - |p-i|\right]$$

$$\bar{x} = M_{10}/M_{00}$$

$$\bar{y} = M_{01}/M_{00}$$

$$\binom{a}{b} = \frac{a!}{b!(a-b)!} \quad \text{binomial coefficient}$$

Calling Procedure: CALL (M,N)

Arguments: N-1 = highest order of moments to be computed.

M = N x N matrix containing the raw moments to be centralized and normalized

$M(p+1, q+1) = M_{pq}$

For output, the raw moments are replaced by the central moments

$$M\ (p+1,\ q+1) = \mu_{pq}$$

```
      [illegible]
      [illegible]
      [illegible]
C
C     [illegible]
C
      [illegible]
      [illegible] )=[illegible]
      [illegible] )=[illegible]
      [illegible]
      [illegible]
      [illegible] )=[illegible]
      [illegible] )=[illegible]
      [illegible]
      K=I-J+1
      [illegible](J,K)=[illegible](J,K-1)+[illegible](J-1,K)
    [illegible] CONTINUE
C
C     [illegible]
C
      [illegible](2,1)/[illegible](1,1)
      Y=[illegible](1,2)/[illegible](1,1)
      [illegible]+[illegible]
      [illegible]=M(1,1)
      [illegible](1,1)=1.
      DO 2 I=2,[illegible]
      JJ=I-1
      DO 2 J=1,JJ
      K=I-J
      [illegible](J,K)=[illegible](J,K)/[illegible]
      DO 2 L=1,JJ
      [illegible]=((L-K+1)+[illegible](L-K-1))/2
      [illegible]=((J+L-1)-[illegible](J-L+1))/2
      DO 2 M=S,T
      [illegible](J,K)=[illegible](J,K)-FL[illegible](S(J-M+1,[illegible]))*FLOAT(S(K-L+M+1,L-1)) (X[illegible](J-M))
     1 (Y[illegible](K-L+1))[illegible](L,L-M)
      KY=S(J-M+1,M)*[illegible](K-L+M+1,L-M)
      LY=J-M
      LY=K-L+1
      L/=L-M
    2 CONTINUE
      RETURN
      END
```

Program Name: MOMENT

Purpose: To calculate up to twentieth order two-dimensional raw moments over an image intensity distribution.

Method: The first Newton-Cotes equation was used.

$$\int_{x1}^{x2} y(x)\ dx = .5h\ (y_1 + y_2) - \frac{1}{12} h^3\ y''$$

The two-dimensional moment M_{pq} is defined in Cartesian coordinates as

$$M_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p\ y^q\ f(x,y)\ dxdy$$

and

$$M_{pq} = \int x^p \left[y^q f(x,y)\, dy \right] dX$$

$$= \int x^p\ V_q\ (x)\ dy$$

where

$$V_q(X) = \int y^q\ f(x,y)\ dy$$

or

$$V_q(X_i) \approx \sum_{j=1} y_j^q\ f_{ji} - \frac{h}{2}\ (y_1^q f_{1i} + y_N^q\ f_{Ni})$$

where f_{ji} is the ji-th element of an N x N image intensity distribution array, y_j is the corresponding vertical coordinate, and h is the width of each image element.

Then

$$M_{pq} \approx \sum_{i=1} x_i^p \, V_q(xi) - \frac{h}{2} \left(x_i^p \, V_{q},_{x=1}, + x_N^p \, V_{q}{'}_{x=N}\right.$$

Define

$$\underline{V} = \begin{bmatrix} V_{q=0,x1} & \cdots & V_{q=0,xN} \\ \vdots & & \vdots \\ V_{q=L,x1} & \cdots & V_{q=L,xN} \end{bmatrix} - \begin{bmatrix} \text{correction} \\ \text{terms} \end{bmatrix}$$

$$= \begin{bmatrix} y_1^0 & \cdots & y_N^0 \\ \vdots & & \vdots \\ y_1^L & \cdots & y_N^L \end{bmatrix} \begin{bmatrix} f_{1,1} & \cdots & f_{1,N} \\ \vdots & & \vdots \\ f_{N,1} & \cdots & f_{N,N} \end{bmatrix} - \begin{bmatrix} \text{correction} \\ \text{terms} \end{bmatrix}$$

$$= \underline{YY}\ \underline{F} - \underline{C}.$$

Then

$$\left[M_{pq}\right] = \underline{V}\ \underline{YY}^T - \underline{C}'$$

$$= (\underline{YY}\ \underline{F} - \underline{C})\ \underline{YY}^T - \underline{C}'$$

Calling Procedure: CALL MOMENT (R, N, M, L)

Arguments: N = number of image elements per row/column of an image intensity distribution array.

R = N x N square matrix containing the values of the image intensity distribution.

L - 1 = highest order of moments to be computed

M = output matrix containing the computed moments

$$M_{pq} = M(p+1,\ q+1)$$

Subroutines Used: VPROD

Note: The image dimension is normalized to unity.

Program Name: INVRNT

Purpose: To calculate moment invariants involving the second and third order central moments.

Method: Direct algebraic computation based on the moment invariants from Ref 1.

Calling Procedure: CALL INVRNT(MI,M,Q)

Arguments: Q-1 = highest order of central moments to be used.

M = Q x Q input matrix of moments to be used to calculate the moment invariants where $M(p+1, q+1) = \mu_{pq}$

MI = one-dimensional output array containing the computed moment invariants

Program Name: MICALC

Purpose: Serves as a monitor program to call appropriate subprograms to input data, compute raw moments, calculate central moments and moment invariants, and then to display them.

Arguments: N = number of image data points per line of a square image

Q-1 = highest calculated moment order

S = number of moment invariants calculated

Image Data: The program inputs imagery data from a tape format. An unformatted READ inputs a vertical image line, 256 pixels per line and 256 lines.

Program Name: VPROD

Purpose: To compute the inner product of two vectors.

$$v = \sum_{i=1}^{N} A_i B_i \ .$$

Method: Double precision multiplication and addition are used; a single precision result is returned. However, the double-precision result is available.

Calling Procedure:

```
CALL VPROD (A,NA,B,NB,N,V)
          or
DOUBLE PRECISION Z, VPROD
Z = VPROD (A,NA,B,NB,N,Z)
```

A - Linear string of elements of the first vector

NA - Interval between elements of A used in the inner product

B - Linear string of elements of the second vector

NB - Interval between elements of B used in the inner product

N - Number of pairs of elements multiplied together

V - Storage location for single precision result

Z - Storage location for double precision result

Error Indicators: None

Subroutines Used: None

```
[illegible]
      [illegible]T([illegible]),[illegible]([illegible]),[illegible]([illegible])
[illegible]
C
C     [illegible]
C
      [illegible]([illegible](J),I=1,[illegible]),J=[illegible],K,[illegible]
      [illegible]
      [illegible]([illegible](I),I=[illegible])
      [illegible]DEL([illegible])=,[illegible],X,12HRI[illegible]=,[illegible],X,
     1[illegible])=,[illegible],X,[illegible](F[illegible])=,[illegible])
C
C     [illegible],V[illegible]L[illegible]Y[illegible]
C     [illegible]
C
      [illegible]K=[illegible]
      [illegible]([illegible](J),[illegible],J=[illegible])
      [illegible]
      CALL [illegible]([illegible])
      [illegible]
    [illegible]T([illegible],[illegible]RAW[illegible]S)
      [illegible],(([illegible](I,JJ),JJ=1,11),II=1,9)
    [illegible]T([illegible],[illegible]2.5)
      [illegible]
      [illegible],(([illegible](I,JJ),JJ=12,9), I=1,9)
    [illegible]T([illegible],[illegible].3)
C
C     [illegible]CH[illegible]T[illegible]AL MOMENTS
C
      CALL C[illegible](F,C)
      [illegible]
    [illegible]T([illegible],[illegible]TRAL MOMENTS)
      [illegible],(([illegible](II,JJ),JJ=1,11),II=1,9)
      [illegible]
      [illegible],(([illegible](II,JJ),JJ=12,9),II=1,9)
C
C     [illegible]CH[illegible]T INVARIANTS
C
      CALL [illegible](C,C,S,[illegible])
      [illegible]
    [illegible]T([illegible],[illegible]INVARIANTS)
      [illegible],([illegible](LL),LL=1,8)
    [illegible]T([illegible].[illegible])
C
C     [illegible]OF [illegible]L[illegible]
```

```
C
      CALL [illegible](  ,  )
C
C       [illegible] INT/[illegible]
C
      CALL [illegible]( 1, )
      GO TO [illegible]
      [illegible]
      [illegible] 111
      [illegible](1H , [illegible] OF FILE)
      STOP
      END
```

Program Name: PPLOT

Purpose: To generate a polar graph on the Calcomp plotter of moment invariants

Calling Procedure: Call (MI,R)

Arguments: R = the number of moment invariants to be plotted

MI = array of R moment invariants

Output generated for on-line Calcomp plotter

Subroutines Used: basic and auxiliary Calcomp routines

Program Name: BPLOT

Purpose: To generate a bar graph on the Calcomp plotter of two-dimensional moments (up to 20th order)

Calling Procedure: CALL BPLOT(M,N)

Arguments: N-1 = highest order of moments to be plotted.

M = N x N matrix containing the moments to be plotted.

$M(p+1,q+1) = M_{pq}$; pq-th moment

Output is generated for on-line Calcomp plotter.

Subroutines Used: basic and auxiliary Calcomp routines

```
      [illegible] F(M,N)
      [illegible] X(21,21),DELX,X[illegible] AY(300),YARRAY(300),[illegible]Y,[illegible]YY,[illegible]YYY,X[illegible],Y[illegible]
      [illegible] M,L,N,[illegible]
      [illegible] Y (301),Y[illegible](301),[illegible] (21),XL[illegible](N),Y[illegible](N),[illegible](N)
C
C     [illegible] AYS FOR MOMENT
C     [illegible] AXIS LABELS
C
      DATA (C[illegible]      /[illegible]       ,[illegible]       ,[illegible]       ,
     1[illegible]              ,[illegible]       ,[illegible]       ,[illegible]       ,
     [illegible]               ,[illegible]       ,[illegible]       ,[illegible]       ,
     [illegible]               ,[illegible]H12    ,[illegible]H13    ,[illegible]       ,
     [illegible]               ,[illegible]       ,[illegible]       ,[illegible]H1[illegible] ,
     [illegible]               ,[illegible]       /
      DATA Y[illegible]/2[illegible]T[illegible]G[illegible]       /
      DATA XL[illegible]/2[illegible]T[illegible]       /
C
C     [illegible] Y [illegible]Y, Y [illegible]Y,
C     [illegible] LABELS FOR PLOT
C
      Y=[illegible].[illegible]
      [illegible]
      DO [illegible] I=1,N
      [illegible]( )-[illegible]
      [illegible](I,[illegible]F[illegible]T(I))
      [illegible]
      Y=(N+1)-J
      Y=Y*[illegible]
      [illegible]X(I)=Y
      [illegible](J)
      [illegible]((J,K))
      [illegible](K)
      [illegible]
      [illegible]
C
C     [illegible]
C     [illegible]
C     [illegible]
C
      [illegible]
      [illegible](2)=[illegible]
      [illegible]
      [illegible]
      [illegible]
      [illegible]
      [illegible]
      [illegible]
      [illegible]
C
C     [illegible]
C
      [illegible]( )=Y[illegible]Y(I+1))/Y[illegible]Y(I+1)
      [illegible]( )=Y[illegible]Y(I+1))/Y[illegible](I+1)
      [illegible]Y [illegible].[illegible]
      [illegible] [illegible].[illegible]
      [illegible] [illegible].[illegible]
```

```
DYYY=DYY+.1
CALL PLOT (XP, [illegible] ,3)
CALL PLOT (X[illegible],YP,2)
CALL [illegible] (X[illegible],Y[illegible],[illegible],-1)
CALL [illegible] (DY,DY,[illegible],Y[illegible]([illegible]),[illegible],2)
CALL [illegible] ([illegible],DYY,[illegible],-1)
CALL [illegible] ([illegible]Y,DYYY,[illegible],Y[illegible]([illegible]),[illegible],0)
[illegible]
CALL PLOT (1)
RETURN
END
```

Program Name: INVRN1

Purpose: To calculate moment invariants

Method: The moment invariants are generated via the group theory procedure developed in Chapter IV. The computed invariants up to the fifth order are listed in Appendix I.

Calling Procedure: CALL INVRN1(MI,M,Q)

Arguments: Q-1 = highest order of central moments (μ_{pq}) to be used in computation. $1 \leq Q \leq 21$.

M = Q x Q input matrix of moments to be used to calculate the moment invariants where

$$M(p+1,\ q+1) = \mu_{pq}$$

MI = one-dimensional cutput array containing the computed moment invariants.

```
      SUBROUTINE INVRN1(XI,M,Q)
      INTEGER Q,QQ,S,T,EXPC,EXPS,B(21,21),A,U,UU,V,VV
      REAL M(Q,Q),X(2,21,21),Y(2,21,21),M1(6,22),III
C
C        GENERATE BINOMIAL COEF MATRIX
C
      QQ=Q+1
      B(1,1)=1
      B(2,1)=1
      B(1,2)=1
      DO 1 I=3,QQ
      A=I-1
      B(I,1)=B(A,1)
      B(1,I)=B(1,A)
      DO 1 J=2,A
      K=I-J+1
      B(J,K)=B(J,K-1)+B(J-1,K)
    1 CONTINUE
C
C        CALC MOMENT VECTOR
C
      DO 13 I=1,2
      DO 13 J=1,21
      DO 13 K=1,21
      X(I,J,K)=0.0
      Y(I,J,K)=0.0
   13 CONTINUE
      X(1,1,1)=2.*M(1,1)
      Y(1,1,1)=0.0
      DO 2 I=1,QQ
      JJJ=1
      IF(I.GT.1)JJJ=2
      DO 2 J=1,JJJ
      K=I-J
      LLL=I-1
      DO 2 L=1,LLL
      S=((L-K+IABS(L-K))/2+1
      T=((J+L-2)-IABS(J-L))/2+1
      DO 2 N=1,T
      COEF=(-1.)**(L+N)*FLOAT(B(J-N+1,N))*FLOAT(B(K-L+N,L-N+1))*M(L,I-L)
       EXPC=K-L+2*(N-1)
       EXPS=L-2*N+1
```

```
      CTEST=.5*FLOAT(EXPC)-AINT(.5*FLOAT(EXPC))
      STEST=.5*FLOAT(EXPS)-AINT(.5*FLOAT(EXPS))
      U=INT(.5*FLOAT(EXPC+1))
      UU=INT(.5*FLOAT(EXPC+2))-INT(.5*(EXPC-1))
      V=INT(.5*FLOAT(EXPS+1))
      VV=INT(.5*FLOAT(EXPS+2))-INT(.5*(EXPS-1))
     13HV=,I ,3X,3HVV=,I )
      COEF=COEF*(.5** (EXPC+EXPS-1))
      IF(STEST.EQ. . )GOTO 1
C
C      CALC COS**EXPC*SIN**EXPS
C      FOR EXPS ODD
C
      IF(EXPC.EQ.0)GOTO 4
      DO 3 II=1,U
      DO 3 JJ=1,V
      COEF1=B(U-II+1,EXPC-U+II+1)*B(V-JJ+1,EXPS-V+JJ+1)*((-1.)**(EXPS-
     1JJ))
      IS=UU+VV+2*(II+JJ-2)+1
      Y(J,K,IS)=Y(J,K,IS)+COEF*COEF1
      ISS=VV-UU+2*(JJ-II)
      IF(ISS.EQ. )GOTO 3
      Y(J,K,IABS(ISS)+1)=Y(J,K,IABS(ISS)+1)+COEF1*COEF*ISS/IABS(ISS)
    3 CONTINUE
      IF(CTEST.EQ..5) GOTO 2
    4 DO 5 JJ=1,V
      COEF3=B(U+1,EXPC-U+1)*B(V-JJ+1,EXPS-V+JJ+1)*(-1.)**(EXPS-JJ)
      IS3=VV+2*(JJ-1)+1
      Y(J,K,IS3)=Y(J,K,IS3)+COEF*COEF3
    5 CONTINUE
      GOTO 2
C
C      CALC COS**EXPC*SIN**EXPS
C      FOR EXPS EVEN
C
    1 IF(EXPC.EQ. )GOTO 14
      IF(EXPS.EQ. )GOTO 1
      DO 11 II=1,U
      DO 11 JJ=1,V
      COEF1=B(U-II+1,EXPC-U+II+1)*B(V-JJ+1,EXPS-V+JJ+1)*((-1.)**(EXPS-
     1JJ))
      IS=UJ+VV+2*(II+JJ-2)+1
      Y(J,K,IS)=Y(J,K,IS)+COEF*COEF1
      ISS=IABS(VV-UU+2*(JJ-II))+1
      Y(J,K,ISS)=Y(J,K,ISS)+COEF*COEF1
   11 CONTINUE
   1  DO 12 II=1,U
      COEF2=B(V+1,EXPS-V+1)*B(U-II+1,EXPC-U+II+1)
      IS2=UU+2*(II-1)+1
      Y(J,K,IS2)=Y(J,K,IS2)+COEF*COEF2
   12 CONTINUE
   1. IF(STEST.EQ..5)GOTO 2
      IF(EXPS.EQ. )GOTO 1
      DO 13 JJ=1,V
      IS3=VV+2*(JJ-1)+1
```

```
      ...?+1,EXP?-U+1)...(V-JJ+1,EXPS-V+JJ+1)*(-1.)**( XPS-JJ)
      Y(J,K,I 3)=X(J,K,153)+COEF*COEF3
  1 3 CONTINUE
  15  COEF4=. *B(U+1,EXPS-U+1)*B(V+1,EXPS-V+1)
      Y(J,K,1)=X(J,K,1)+COEF*COEF4
    2 CONTINUE
      DO 19 I=2,??
      JJ=2
      IF(I.LE.?)JJ=1
      DO 19 J=1,JJ
      K=I-J
      LL=K-J+1
      CTEST=. *FLOAT(I)-AINT(.5*FLOAT(I))
      LLL=I-K+J-1
      IF(CTEST.EQ..5)LLL=2
      DO 19 L=LLL,LL,2
      PRINT 1 2,X(J,K,L),Y(J,K,L),J,K,L
  1 2 FORMAT(1H ,2F1 .4,5X,3I )
  19  CONTINUE
C
C        CALC INV?NTS FROM VECTORS
C
      N=1
      DO 17 I=1,?
      DO 17 J=I,?,2
      KK=J+1
      DO 17 K=KK,??,2
      LL=2
      IF((KK.EQ.K).OR.(I.EQ.1))LL=1
      DO 17 L=1,LL
      ?I(N)=Y(1,J,?) Y(L,K-L,?)+Y(1,J,?) Y(L,K-L,I)
      N=N+1
  17  CONTINUE
      S=(?-1) 2
      PRINT 1
  1   FORMAT(1H ,  ? INVARIANTS)
      PRINT 1 1,(?I(I),I=1,S)
  1 1 FORMAT(1H ,  3.?)
      RETURN
      END
```

Program Name: PARRAY

Purpose: To serve as a monitor program to generate plots of moments as a function of threshold.

Calling Procedure: CALL PARRAY (M, THRSH,A,B,C,D,E,F)

Arguments:

M =	20 x 4 x 4 input matrix containing the moments to be plotted
THRSH =	1 x 22 array of threshold values
A,B,C,D =	image parameters; roll, pitch, yaw, range
E =	maximum image intensity
F =	minimum image intensity

Subroutines Used: LPLOT

```
      [illegible] (X[illegible],F)
      [illegible] Y(12),[illegible](12),LA[illegible](1[illegible])
      DATA L[illegible] [illegible]
     1[illegible],
     2[illegible] /
      L=1
      DO 4 J=1,[illegible]
      DO 4 K=[illegible],J
      [illegible]=J+[illegible]
      DO 2 I=1,[illegible]
      [illegible] Y(I)=[illegible](I,K,[illegible])
    2 CONTINUE
      [illegible](L)
      CALL [illegible](Y[illegible]X,[illegible],F)
      IF(L.EQ.[illegible]) CALL [illegible]
      L=L+1
    4 CONTINUE
      CALL [illegible]
      RETURN
      END
```

Program Name: MIARAY

Purpose: To serve as a monitor program to generate plots of moment invariants as a function of threshold.

Calling Procedure: CALL MIARAY (MII,THRSH,A,B,C,D,E,F,)

Arguments:

MII =	20 x 7 input matrix of the moment invariants to be plotted.
THRSH =	1 x 22 array of threshold values
A,B,C,D,=	image parameters, roll, pitch, yaw, range
E =	maximum image intensity
F =	minimum image intensity

Subroutines used: LPLOT

```
[illegible] Y( [illegible] ,THRC[illegible], [illegible] ,F)
[illegible] Y( [illegible] ), [illegible] ),TH[illegible]( [illegible] ),L[illegible]( [illegible] )
[illegible] [illegible] [illegible] ,
[illegible] [illegible] [illegible] /
[illegible] J=1, [illegible]
[illegible] 2 I=1, [illegible]
[illegible] Y( [illegible] )=[illegible]( [illegible] , [illegible] )
2 CONTINUE
[illegible]=[illegible](J)
[illegible] [illegible](Y[illegible]AY,TH[illegible],X,[illegible],[illegible],[illegible],[illegible],[illegible],F)
[illegible](J.[illegible]. )[illegible]( [illegible] )
[illegible] CONTINUE
[illegible] [illegible](2H[illegible])
RETURN
END
```

Program Name: IMAGE

Purpose: To generate a character printed image from imagery data.

Method: The imagery data is read into an array, and the minimum and maximum intensities are found. This range of image intensities is divided into 10 levels. Each level is assigned a character: __, 1, 2, 3,···9, from minimum to maximum, respectively. The image is gated to narrow the viewing area. The gated array is then character printed on the line printer.

Subroutines used: None

```
      [illegible]
      [illegible] (2), ( [illegible] ), [illegible] X,[illegible]
C
C          READ IN HEADER AND IMAGE DATA
C
      DO 1 I= ,1
      DO 2 J= ,
      READ(1)(HEAD(N),N=[illegible]),KDIM,JDIM,(HEAD(N),N=11,12)
      READ (1)((I(II,KK),II=1,JDIM),KK=1,KDIM)
    2 CONTINUE
C
C      PRINT HEADER DATA
C
      PRINT [illegible]
      FORMAT([illegible],7X, [illegible]V,[illegible]X,[illegible]HOFFIDOT,3X,[illegible]HOFFSTEL,[illegible]X,[illegible]HOFROLL,[illegible]X,
     1 HOP[illegible]CH,[illegible]X, [illegible]YAW,[illegible]X,[illegible]ROLL,[illegible]X,[illegible]HTPITCH,[illegible]X,[illegible]HTYAW, X,[illegible]RANGE,
     2 X,[illegible]HOFIDOT, X,[illegible]HOFSTEL, X,[illegible]HKDIM,[illegible]X,[illegible]HJDIM)
      PRINT [illegible] ,(HEAD(K),K=[illegible],12),KDIM,JDIM
    1 FORMAT([illegible] ,[illegible] ,[illegible])
C
C      CALCULATE MIN/MAX
C
      [illegible]
      [illegible]X=[illegible]
      [illegible] L=[illegible],J[illegible]
      DO [illegible] M=[illegible],KD[illegible]
      IF(I(L,[illegible]) [illegible] (L,M)
      IF(I(L,[illegible]) [illegible] (L,M)
      CONTINUE
C
C      [illegible]
C
      [illegible]
     1 [illegible]
      [illegible]=( [illegible] )/ [illegible]
C
C      [illegible]
C
      MIN=JDIM
      [illegible]X=[illegible]
      [illegible]MIN=KD[illegible]
      JMAX=[illegible]
```

```
DO [illegible] K=1,JDIM
DO [illegible] M=1,KDIM
IF(R(IK,IM).LT.(MIN+1.0))GOTO [illegible]
IF (IK.LT.IMIN) IMIN=IK
IF (IK.GT.IMAX) IMAX=IK
IF (IM.LT.JMIN) JMIN=IM
IF (IM.GT.JMAX) JMAX=IM
CONTINUE

PRINT [illegible],IMIN,IMAX,JMIN,JMAX
FORMAT([illegible],[illegible]X,[illegible] POSSIBLE,[illegible]X,[illegible]HIMIN=,I[illegible],[illegible]X,[illegible]HIMAX=,I1[illegible],[illegible]X,
1 [illegible]HJMIN=,I[illegible],[illegible]X,[illegible]HJMAX=,I[illegible])
DO [illegible] LL=IMIN,IMAX
DO [illegible] MM=JMIN,JMAX
CHAR=A
IF(R(LL,MM).LT.(MAX-[illegible].[illegible]INC))CHAR=B
IF(R(LL,MM).LT.(MAX-[illegible].[illegible]INC))CHAR=C
IF(R(LL,MM).LT.(MAX-[illegible].[illegible]INC))CHAR=D
IF(R(LL,MM).LT.(MAX-[illegible].[illegible]INC))CHAR=E
IF(R(LL,MM).LT.(MAX-[illegible].[illegible]INC))CHAR=F
IF(R(LL,MM).LT.(MAX-[illegible].[illegible]INC))CHAR=G
IF(R(LL,MM).LT.(MAX-[illegible].[illegible]INC))CHAR=H
IF(R(LL,MM).LT.(MAX-[illegible].[illegible]INC))CHAR=S
IF(R(LL,MM).LT.(MAX-[illegible].[illegible]INC))CHAR=I
[illegible](LL,MM)=CHAR
CONTINUE
DO [illegible] [illegible]=IMIN,IMAX
PRINT [illegible],([illegible]),[illegible]=[illegible],[illegible]Y)
FORMAT([illegible]H,[illegible])
CONTINUE
[illegible]
END
```

Program Name: LPLOT

Purpose: To generate a line plot of moments or moment invariants as a function of threshold.

Calling Procedure: CALL LPLOT (YARRAY,XARRAY,LABEL, R,P,Y,RG,MX,MN)

Arguments:

YARRAY = one-dimensional array of abcissa values

XARRAY = one-dimensional array of threshold valves

LABEL = label of abcissa

R = target roll

P = target pitch

Y = target yaw

RG = target range

MX = maximum image intensity

MN = minimum image intensity

Program Name: THRSHD

Purpose: To determine a threshold level to suppress the background from imagery data

Method: The second and third order raw and central moments and the corresponding moment invariants are computed as a function of threshold level where

THRSHD = MIN ~ constant · (MAX-MIN)

MAX = maximum image intensity

MIN = minimum image intensity

Subprograms Used: CENTER

INVRNT

PARRAY

MIARAY

VITA

Tyle T. Kanazawa was born on 8 January 1957 in Greenville, Texas. He graduated from Texas Tech University in 1979 with a Bachelor of Science degree in Electrical Engineering. His commission was obtained in May 1979 through the Air Force ROTC program. He entered the Air Force Institute of Technology, Wright-Patterson AFB, in June 1979 to pursue a Master of Science in the Electro-optics program.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This thesis investigates the application of two-dimensional moment invariants to image pattern recognition. The general problem studied is how to identify an aircraft target and its orientation in real time. The method of moment invariants provides a clever feature extraction technique to reduce the information in an image to a finite number of quantities which are translation, size, and rotation independent. Most of the previous work on image pattern

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recognition has been based on the results obtained by M.K. Hu, who relied on the theory of algebraic invariants. In this thesis, a set of moment invariants is derived from the group-theoretical properties of the two-dimensional rotation group applied to the moments of an image intensity function. It is shown that Hu's invariants can be obtained from this set and is, in fact, an equivalent complete description of the image. The application of group methods to moments presents a general procedure for calculating moment invariants under any linear transformation. The image signal effect of thresholding the background clutter is also discussed.

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